A Realist Perspective on Proton Pressure

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PRD111 (2025) 034047, arxiv:25xx.xxxxx

Jefferson Lab

Pressure in the proton

- Pressure in the proton has become a hot topic.
- Empirical extractions happening at JLab!
  - Burkert, Elouadrhiri & Girod, Nature (2018)
  - Burkert, Elouadrhiri & Girod, 2104.02031
  - Duran &al., Nature (2023)
- But what does this pressure really mean?
  - The interpretation is somewhat controversial.
  - I'll argue that **it really is pressure**.
  - ...& comes from a mix of forces & particle motion.

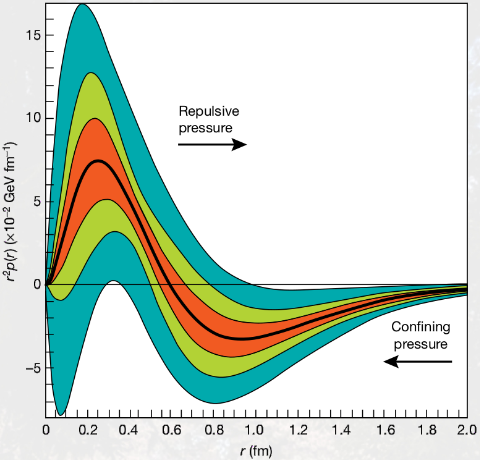


Figure: Burkert, Elouadrhiri & Girod, Nature (2018)


The Energy-Momentum Tensor

The energy-momentum tensor

- The energy-momentum tensor describes **density** and **flow** of energy & momentum.
- Also known as the stress-energy tensor.

Energy density

Momentum densities

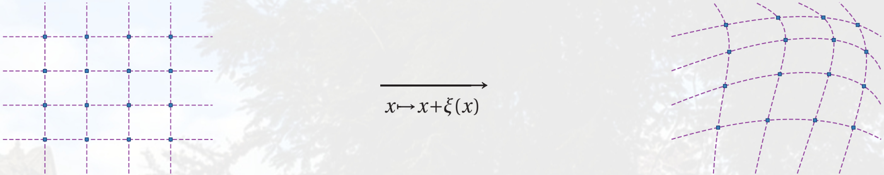
$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Energy fluxes

Stress tensor

Noether's theorems and spacetime distortions

- Conserved current from *local* spacetime translations (**Noether's second theorem**):



  - **Noether's theorems**: symmetries imply conservation laws
  - *Local* translation: move spacetime differently everywhere

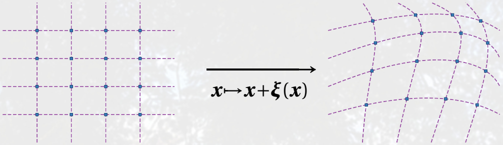
- The **energy-momentum tensor** quantifies response to these deformations

$$\delta_\xi S_{\mathrm{QCD}} = \int \mathrm{d}^4 x \, T^{\mu\nu}_{\mathrm{QCD}}(x) \partial_\mu \xi_\nu(x)$$

  - Conserved if the action is invariant

- 3×3 sub-matrix of the energy-momentum tensor.

$$\begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$



Stress tensor

- How the QCD action responds to *spatial* reconfiguration of the fields:

$$\delta_\xi S_{\mathrm{QCD}} = -\int \mathrm{d}^4x \, (\nabla_i \xi^j(\boldsymbol{x})) \, T^{ij}(x)$$

  - **Stress**: internal forces resisting *spatial* deformation
  - Deformations are a virtual pressure gauge

Lorcé, Metz, Pasquini & Rodini, JHEP 11 (2021) 121

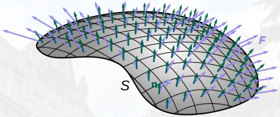
AF, 2506.04510

Stress & momentum flux density

- Continuity equation:

$$\partial_\mu T^{\mu\nu}(x) = 0$$

  - Energy/momentum transmitted locally.



Momentum densities

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Stress tensor

- Integral form for spatial components:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_V \mathrm{d}^3x\, T^{0\nu}(\boldsymbol{x},t)\right]}_{\text{Energy/momentum in region}} = -\underbrace{\oint_{\partial V} \mathrm{d}S\, \hat{n}_i\, T^{i\nu}(\boldsymbol{x},t)}_{\text{Flux out of region}}$$

  - Stress tensor tells us how momentum enters or leaves a region
  - Can happen through particle flow or force transmission
  - (Still has a valid stress/pressure interpretation either way)

Particle flux

$$T^{ij}(\boldsymbol{x}) = v^i p^j \delta^{(3)}(\boldsymbol{x} - \boldsymbol{q})$$

Force

$$T^{ij}(\boldsymbol{x}) = \hat{n}^i \hat{n}^j \frac{F}{A}$$

- Momentum can enter/leave region because particles enter/leave.
  - Strictly positive contribution to stress.
- Momentum can enter/leave region because of forces.
  - Can produce positive or negative stress.
- Stress tensor includes both.
  - Is the hadronic stress tensor due to motion or forces? (Or both?)

Positive or negative stress?

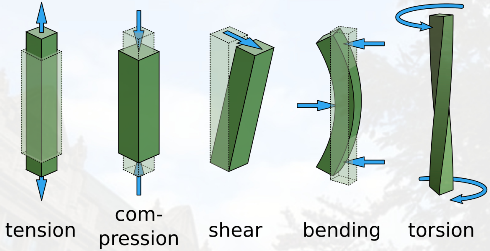
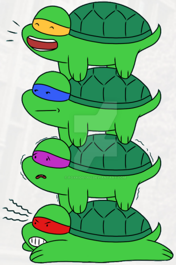


Image: MikeRun (Wikimedia), modified

- Negative stress / negative "pressure"
- Tension / pulling / stretching
- More tension on highest spring

$$\nabla_i T^{ij} = -\rho g \hat{z}^j = f^j_{\text{grav}}$$



- Positive stress / positive "pressure"
- Compression / pushing / squishing
- More compression at bottom of pile

$$\nabla_i T^{ij} = -\rho g \hat{z}^j = f^j_{\text{grav}}$$

Sign of stress unrelated to attraction/repulsion in force law!

Turtle image by FoxKids1302 (DeviantArt)

Cauchy's first law of motion

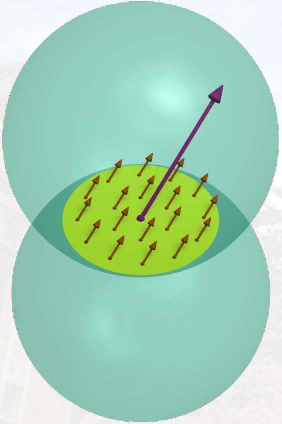


Image: Jorge Stolfi (Wikimedia)

- **Cauchy's first law of motion**: for an *open* system:

$$\frac{\partial T^{0j}(\boldsymbol{x}, t)}{\partial t} = f^j_{\text{net}}(\boldsymbol{x}, t) = f^j_{\text{external}}(\boldsymbol{x}, t) - \nabla_i T^{ij}(\boldsymbol{x}, t)$$

- Stress tensor tells us how system responds to external forces.
  - **Important**: external is force not part of the stress tensor!
  - Stresses are fluxes between adjacent cells within system.

- Works for fluids, solids, liquid crystals, neutron stars, ...

- Can this be applied to quantum systems?


Realist Quantum Mechanics

Observer-independent quantum mechanics

- Vanilla quantum mechanics is an algorithm to calculate event rates.
  - Agnostic about what happens outside of measurements.

- Mechanical properties assume some underlying reality.
  - Interpretation assumes quarks & gluons are moving & exerting forces

- Multiple realist formulations of quantum mechanics exist:
  - Everett's many-worlds interpretation
  - Ghirardi–Rimini–Weber collapse theory
  - de Broglie-Bohm pilot wave theory

- What they have in common: **wave function realism**
  - **PBR theorem** makes this necessary.
  - Pusey, Barrett & Rudolph, Nature Phys 8 (2012) 476

Wave function as an inviscid fluid

- Using a polar form for the wave function:

$$\psi(\boldsymbol{x},t) = \mathscr{R}(\boldsymbol{x},t)\,\mathrm{e}^{i\mathscr{S}(\boldsymbol{x},t)/\hbar} \qquad \rho(\boldsymbol{x},t) = |\psi(\boldsymbol{x},t)|^2 \qquad \boldsymbol{v}(\boldsymbol{x},t) = \frac{\boldsymbol{\nabla}\mathscr{S}(\boldsymbol{x},t)}{m},$$

- Schrödinger equation $\Longrightarrow$ **Navier-Stokes equation** (without viscosity):

$$\frac{\partial}{\partial t}\left[\boldsymbol{v}\rho\right] + \boldsymbol{\nabla}\cdot\left[\rho\,\boldsymbol{v}\otimes\boldsymbol{v} + \boldsymbol{T}_Q\right] = 0$$

  - Suggests hydrodynamic interpretation of wave function.
  - E. Madelung, Naturwissenschaften 14 (1926) 1004, Z. Phys 40 (1927) 322

- The **quantum stress tensor** appearing here:

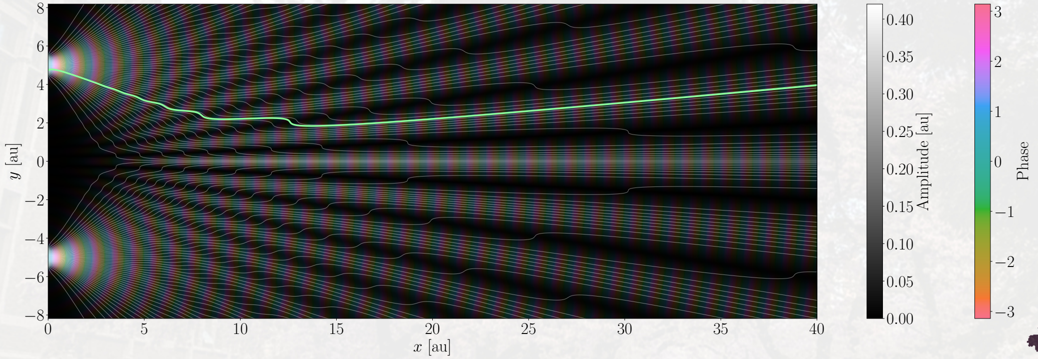
$$T_Q^{ij} = \frac{\hbar^2}{2m}\Big((\nabla_i\mathscr{R})(\nabla_j\mathscr{R}) - \mathscr{R}(\nabla_j\nabla_i\mathscr{R})\Big)$$

  - T. Takabayasi, Prog. Theor. Phys. 8 (1952) 143

- **Pilot wave theory** can help clarify the meaning of this.

The pilot wave interpretation

- The **pilot wave interpretation** ascribes positions *and* wave functions to particles.
  - Discovered by de Broglie in 1927 (J Phys Rad 8), rediscovered by Bohm in 1952 (Phys Rev 85).
  - Also known as **Bohmian mechanics**.
- Measurements reveal pre-existing particle positions.
  - All other observables (momentum, spin, etc.) are created by the measurement process.
- A pilot wave (the wave function) guides particle motion.



Polar decomposition of the Schrödinger equation

- Pilot wave formulation derived from **polar decomposition** of the wave function:

$$\psi(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n,t) = \mathscr{R}(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n,t)\,\mathrm{e}^{i\mathscr{S}(\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n,t)/\hbar}$$

- Schrödinger equation becomes coupled equations for real-valued fields.
- **Hamilton-Jacobi equation** for phase function:

$$\frac{\partial\mathscr{S}}{\partial t} + \sum_{\text{particles}} \frac{(\nabla_n\mathscr{S})^2}{2m_n} + V - \sum_{\text{particles}} \frac{\hbar^2}{2m_n}\frac{\nabla_n^2\mathscr{R}}{\mathscr{R}} = 0$$

  - Suggests particles carry momenta $\boldsymbol{p}_n = \nabla_n\mathscr{S}(\boldsymbol{q},t)$ when configuration is $\boldsymbol{q} = (\boldsymbol{q}_1,\ldots,\boldsymbol{q}_N)$.
- **Continuity equation** for modulus function:

$$\frac{\partial\mathscr{R}^2}{\partial t} + \sum_{\text{particles}} \nabla_n\cdot\left[\frac{(\nabla_n\mathscr{S})}{m_n}\mathscr{R}^2\right] = 0$$

  - Suggests probability distribution given by $\mathscr{R}^2 = |\psi|^2$.
  - Suggests particles have velocity $\dot{\boldsymbol{q}}_n = \nabla_n\mathscr{S}/m_n$

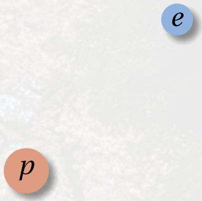
Momentum in the pilot wave formulation

- Pilot wave formulation ascribes momentum to particles:

$$\boldsymbol{p}_{\mathrm{B}}^{(n)} = \boldsymbol{\nabla}_n \mathscr{S} = -\frac{i}{2}\,\frac{\psi^* \overleftrightarrow{\boldsymbol{\nabla}}_n \psi}{\psi^* \psi}$$

  - This is a **hidden variable**.
  - $\boldsymbol{p}_{\mathrm{B}}^{(n)}$ is **not** usually the measured canonical momentum.
  - Canonical momentum created by measurements, which alter the system.
  - Exception: $\boldsymbol{p}_{\mathrm{B}}^{(n)}$ is the canonical momentum **for momentum eigenstates**.

- For real wave functions, $\boldsymbol{p}_{\mathrm{B}}^{(n)} = 0$.
  - Includes hydrogen ground state.
  - Force from **quantum potential** keeps electron suspended.
  - Animated figure on right.



Quantum potential, quantum force and quantum stress

- Particles evolve under effective Hamiltonian:

$$H_{\text{eff}} = -\frac{\partial \mathscr{S}}{\partial t} = \sum_{\text{particles}} \frac{\boldsymbol{p}_n^2}{2m_n} + V + Q$$

- Classical equation modified by a **quantum potential**:

$$Q = -\sum_{\text{particles}} \frac{\hbar^2}{2m_n} \frac{\boldsymbol{\nabla}_n^2 \mathscr{R}}{\mathscr{R}} = -\sum_{\text{particles}} \frac{\hbar^2}{2m_n} \frac{\boldsymbol{\nabla}_n^2 |\psi|}{|\psi|}$$

  - Responsible for interference, tunneling, entanglement, etc.

- Quantum force related to quantum stress tensor!

$$-\nabla_i T_Q^{ij}(\boldsymbol{x}, t) = -\rho(\boldsymbol{x}, t) \nabla^j \Bigg[\underbrace{-\frac{\hbar^2}{2m} \frac{\nabla^2 \mathscr{R}(\boldsymbol{x}, t)}{\mathscr{R}(\boldsymbol{x}, t)}}_{\text{quantum potential}} \Bigg] = f^j_{\text{pilot}}(\boldsymbol{x}, t)$$

  - Force exerted by wave function on the particle


Example: Hydrogen Atom

Stress densities: non-relativistic

$$\langle\Psi|\hat{T}^{ij}_f(\boldsymbol{x},t)|\Psi\rangle = \int \mathrm{d}^3R \Bigg\{ \underbrace{-\frac{\Psi^*(\boldsymbol{R},t)\overleftrightarrow{\nabla}^i\overleftrightarrow{\nabla}^j\Psi(\boldsymbol{R},t)}{4M}\mathfrak{a}_f(\boldsymbol{x}-\boldsymbol{R})}_{\text{Dynamic stress}} + \underbrace{\Psi^*(\boldsymbol{R},t)\Psi(\boldsymbol{R},t)\mathfrak{t}^{ij}_f(\boldsymbol{x}-\boldsymbol{R})}_{\text{Internal stress}} \Bigg\}$$

- **Dynamic stress** due to barycentric motion and wave packet dispersion.
  - Includes the quantum stress.
- **Internal stress** due to internal motions or forces.
  - Smeared by barycenter probability density.
  - Conveniently related to the Breit frame Fourier transform:

$$\mathfrak{t}^{ij}_f(\boldsymbol{b}) = \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\left(\frac{\boldsymbol{\Delta}^i\boldsymbol{\Delta}^j - \delta^{ij}\boldsymbol{\Delta}^2}{4M}D_f(\boldsymbol{\Delta}^2) - M\delta^{ij}\bar{c}_f(\boldsymbol{\Delta}^2)\right)\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}}$$

- **Strictly non-relativistic** breakdown.
  - Afforded by Galilei symmetry (absoluteness of simultaneity).

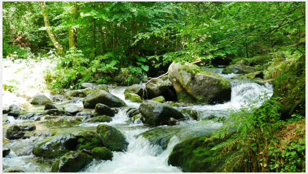


Image: dynamic pressure from water flow

$$t_a^{ij}(\boldsymbol{b}) = \left(\frac{m_a \mathscr{R}(\boldsymbol{r})}{\mu}\right)^2 \left\{ \underbrace{\frac{k_{\mathrm{B}}^i(\boldsymbol{r}) k_{\mathrm{B}}^j(\boldsymbol{r})}{\mu}}_{\text{Particle flux}} + \underbrace{\frac{\hbar^2}{2\mu \mathscr{R}(\boldsymbol{r})^2} \Big[(\nabla^i \mathscr{R}(\boldsymbol{r}))(\nabla^j \mathscr{R}(\boldsymbol{r})) - \mathscr{R}(\boldsymbol{r})(\nabla^i \nabla^j \mathscr{R}(\boldsymbol{r})) \Big]}_{\text{Quantum stress}} \right\} \Bigg|_{\boldsymbol{r}=m_a \boldsymbol{b}/\mu}$$

- Apply polar decomposition to *internal* wave function:

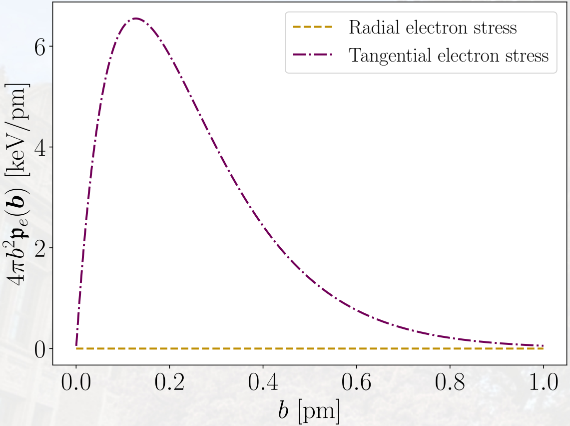
$$\psi(\boldsymbol{r}, t) = \mathscr{R}(\boldsymbol{r})\, \mathrm{e}^{i\mathscr{S}(\boldsymbol{r},t)}$$

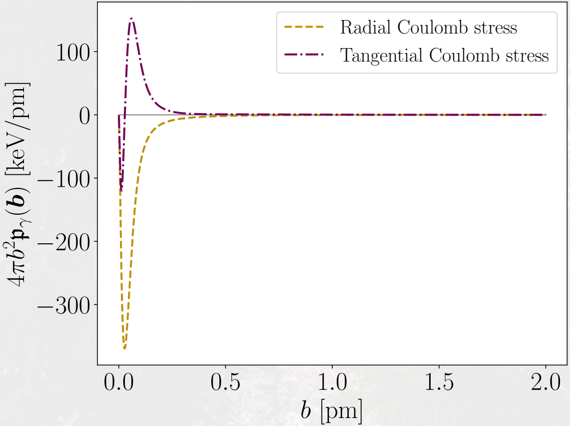
- **Particle flux** related to orbital angular momentum:

$$\boldsymbol{k}_{\mathrm{B}}(\boldsymbol{r}) = \frac{m_l \hat{\boldsymbol{\phi}}}{r \sin\theta}$$

- **Quantum stress** keeps stationary states stable.
  - Keeps $m_l = 0$ states static — cancels binding force!
  - Keeps $m_l \neq 0$ states in circular orbits.

Stresses in hydrogen atom





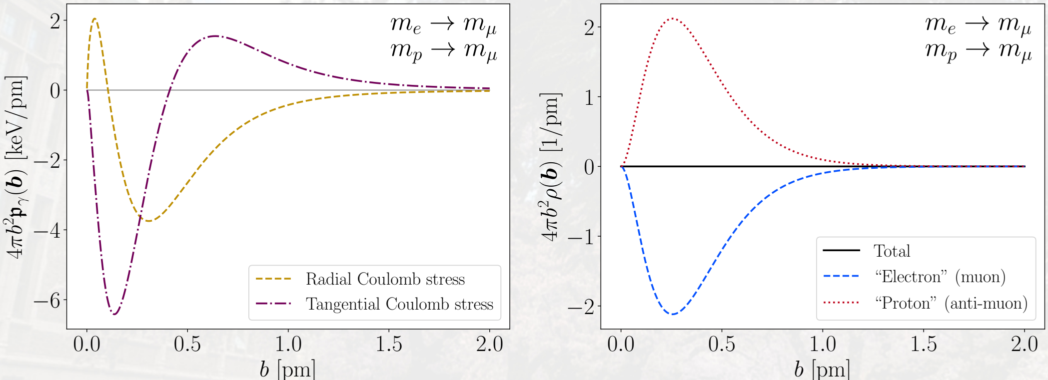
- Particle stress is positive & **tangential**.
  - $l = 0$, so it **really is stress**.
- Tension in Coulomb field keeps the system together.

Coulomb stress as an ensemble average

- Hydrogen atom is not a classical source!

$$\left\langle -\hat{E}^i\hat{E}^j + \frac{1}{2}\delta^{ij}\hat{\boldsymbol{E}}^2 \right\rangle_\psi \neq -\langle \hat{E}^i\rangle_\psi \langle \hat{E}^j\rangle_\psi + \frac{1}{2}\delta^{ij}\langle \hat{\boldsymbol{E}}\rangle_\psi^2$$

- Coulomb stress looks like **ensemble average** of two point sources.
  - Does *not* look like a shell of charge.
  - Mass-balanced system (setting $m_p = m_e$) can illustrate this more starkly.



Revisiting Cauchy's first law

- Apply Cauchy's first law to electron in the hydrogen atom?

$$f^j_{\text{external}}(\boldsymbol{x}) = \nabla_i \langle T^{ij}_e(\boldsymbol{x})\rangle_\psi$$

  - Applies to static systems—okay for hydrogen ground state!
- In terms of mechanical form factors:

$$f^j_{\text{external}}(\boldsymbol{b}) = -M\nabla^j \int \frac{\mathrm{d}^3\boldsymbol{\Delta}}{(2\pi)^3}\bar{c}_e(\boldsymbol{\Delta}^2)\,\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{b}} = \underbrace{\frac{(\alpha m_e)^3}{\pi}\mathrm{e}^{-2\alpha m_e b}}_{\text{Probability density}} \times \underbrace{\frac{-\alpha\hat{b}}{(m_e b/\mu)^2}}_{\text{Coulomb force}}$$

  - AF, PRD111 (2025) 034047
- Electron stress tells us how electron responds to external forces.
  - Information about the Coulomb force is hidden in it.
- If there's a relativistic generalization too, $\bar{c}_f(\boldsymbol{\Delta}^2)$ could map QCD forces!
  - Polyakov & Son, JHEP 09 (2018) 156
  - Won, Kim & Kim, JHEP 05 (2024) 173
  - Ji & Yang, 2503.01991


Relativistic Stress Densities

Relativistic convolution

- Moving system is distorted by Lorentz boosts:

$$\langle\Psi|\hat{T}^{\mu\nu}(\boldsymbol{x},t)|\Psi\rangle = \int \mathrm{d}^3R \left\{\rho(\boldsymbol{R},t)\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\mathfrak{t}^{\alpha\beta}\left(\Lambda^{-1}[x-R]\right) + T_Q^{\mu\nu}(\boldsymbol{R},t)\mathfrak{a}_f\left(\Lambda^{-1}[x-R]\right)\right\}$$

  - Λ is a boost by **Bohmian velocity**
  - $\rho(\boldsymbol{R},t)$ is probability density for barycenter
  - They boost & smear the **internal energy-momentum tensor** $\mathfrak{t}^{\alpha\beta}$
  - There's also a quantum stress-energy $T_Q^{\mu\nu}$
- Mechanical (gravitational) form factor breakdown:

$$\langle p'|\hat{T}_f^{\mu\nu}(0)|p\rangle = 2P^{\mu}P^{\nu}A_f(\Delta^2) + \frac{\Delta^{\mu}\Delta^{\nu} - \Delta^2\eta^{\mu\nu}}{2}D_f(\Delta^2) + 2m^2\eta^{\mu\nu}\bar{c}_f(\Delta^2)$$

- Goal: solve for $\mathfrak{t}^{\alpha\beta}$ in terms of the form factors.
  - Technical caveat: it's only solvable for acceleration-free flow
  - No rigid bodies in relativity—acceleration deforms structure

- Polar decomposition for spin-zero wave function:

$$\psi(\boldsymbol{r},t) \equiv \langle 0|\hat{\phi}(\boldsymbol{r},t)|\Psi\rangle = \mathscr{R}(\boldsymbol{r},t)\,\mathrm{e}^{i\mathscr{S}(\boldsymbol{r},t)}$$

- Klein-Gordon equation becomes coupled equations for real-valued fields.

- **Hamilton-Jacobi equation**:

$$\left(-\frac{\partial\mathscr{S}}{\partial t}\right)^2 = m^2 + (\nabla\mathscr{S})^2 + \frac{\partial^2\mathscr{R}}{\mathscr{R}}$$

- **Bohmian energy-momentum**:

$$p_{\mathrm{B}}^{\mu}(\boldsymbol{r},t) = \partial^{\mu}\mathscr{S}(\boldsymbol{r},t)$$

- **Continuity equation**:

$$\partial_{\mu}\left[(\partial^{\nu}\mathscr{S}(\boldsymbol{r},t))\mathscr{R}^2(\boldsymbol{r},t)\right] = 0$$

- **Probability density**:

$$\rho(\boldsymbol{r},t) = -2\big(\partial_t\mathscr{S}(\boldsymbol{r},t)\big)\mathscr{R}^2(\boldsymbol{r},t)$$

- **Bohmian velocity**:

$$\boldsymbol{v}(\boldsymbol{r},t) = -\frac{\nabla\mathscr{S}(\boldsymbol{r},t)}{\partial_t\mathscr{S}(\boldsymbol{r},t)} = -\frac{\psi^*(\boldsymbol{r},t)\overleftrightarrow{\nabla}\psi(\boldsymbol{r},t)}{\psi^*(\boldsymbol{r},t)\overleftrightarrow{\partial}_t\psi(\boldsymbol{r},t)}$$

Stress densities: relativistic

$$\langle\Psi|\hat{T}^{\mu\nu}(\boldsymbol{x},t)|\Psi\rangle = \int \mathrm{d}^3R\,\Big\{\underbrace{\rho(\boldsymbol{R},t)\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}}_{\text{Boost \& smearing}}\overbrace{\mathfrak{t}^{\alpha\beta}\big(\Lambda^{-1}[x-R]\big)}^{\text{Internal stress-energy}} + \underbrace{T_Q^{\mu\nu}(\boldsymbol{R},t)}_{\text{Quantum stress-energy}}\mathfrak{a}_f\big(\Lambda^{-1}[x-R]\big)\Big\}$$

- Quantum stress-energy tensor:

$$T_Q^{\mu\nu}(\boldsymbol{r},t) = \big(\partial^\mu\mathscr{R}(\boldsymbol{r},t)\big)\big(\partial^\nu\mathscr{R}(\boldsymbol{r},t)\big) - \mathscr{R}(\boldsymbol{r},t)\big(\partial^\mu\partial^\nu\mathscr{R}(\boldsymbol{r},t)\big)$$

  - Smeared by Fourier transform of A_f.

- Internal energy density:

$$\mathfrak{t}_f^{00}(\boldsymbol{x}) = m\int\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\left(A_f(-\boldsymbol{\Delta}^2) + \frac{\boldsymbol{\Delta}^2}{4m^2}D_f(-\boldsymbol{\Delta}^2) + \bar{c}_f(-\boldsymbol{\Delta}^2)\right)\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{x}}$$

- **Internal stress tensor**:

$$\mathfrak{t}_f^{ij}(\boldsymbol{x}) = m\int\frac{\mathrm{d}^3\Delta}{(2\pi)^3}\left\{\frac{\Delta^i\Delta^j - \boldsymbol{\Delta}^2\delta^{ij}}{4m^2}D_f(-\boldsymbol{\Delta}^2) - \delta^{ij}\bar{c}_f(-\boldsymbol{\Delta}^2)\right\}\mathrm{e}^{-i\boldsymbol{\Delta}\cdot\boldsymbol{x}}$$


Wrapping Up

Conclusions & Outlook

- Realist formulations of quantum mechanics are helpful!
  - Allow causal & ontological thinking about sub-atomic systems.
  - Mechanical properties make sense in such formulations.
- Can also formulate a relativistic convolution formalism for densities.
  - Allows obtaining 3D relativistic spatial densities with a probability interpretation!
  - Requires using Bohmian rather than canonical momentum.
- Application to relativistic fermions (such as the proton) underway.
  - Hydrodynamic formulation worked out by Takabayasi, Prog Theor Phys Supp 4 (1957) 1
  - Sadly, I didn't have results ready in time for this talk.
- Recommended reading (if you're interested in pilot wave theory):
  - J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics*
  - David Bohm and Basil Hiley, *The Undivided Universe*
  - Tim Maudlin, *Quantum Non-Locality and Relativity*

Thank you for your time!


Backup Slides

Hidden variables and Bell-type inequalities

- **Common myth**: Bell's theorem rules out hidden variables
- **Correction**: Bell's theorem shows quantum mechanics is (in some sense) non-local
  - Bell and CHSH inequalities are derived assuming only local causality and detector setting independence—*not* hidden variables or determinism!
- In what sense though?
  - Wave function lives in configuration space
  - Its evolution generally depends holistically on entire experiment (entanglement!)
  - QFT is local in a weaker sense (cluster decomposition, no signalling, canonical commutation relations) ...but *is still Bell non-local*
- More information:
  - J.S. Bell, "The Theory of Local Beables"
    - Contained in *Speakable and Unspeakable in Quantum Mechanics*
    - Probably the most succinct account
  - Tim Maudlin, *Quantum Non-Locality and Relativity*
    - By far the most comprehensive account

Experimental refutation of pilot wave theory?

- Sharoglazova *et al.* (Nature 643 (2025) 67) claims to experimentally challenge pilot wave theory
  - Quantum tunneling experiment with photons in waveguides
  - Finds zero Bohmian velocity (i.e., zero phase gradient $\boldsymbol{\nabla}\mathscr{S}$)

- But Sharoglazova *et al.* set up a **static scenario**:
  - Fully reflected standing wave
  - Constant-phase evanescent wave
  - These findings *would not happen with a net probability flux*

- Vanilla QM has **zero probability current** in static systems

$$\boldsymbol{v}_{\mathrm{B}}(\boldsymbol{x},t) = \frac{\boldsymbol{j}(\boldsymbol{x},t)}{\rho(\boldsymbol{x},t)} \xrightarrow[\text{static system}]{} 0$$

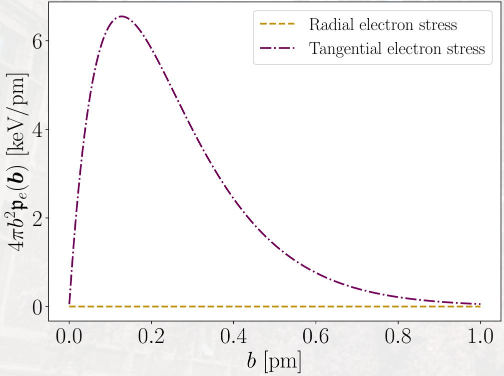
  - Their Bohmian velocity finding is actually sensible, & doesn't refute pilot wave theory
  - Finding *really* refutes naive picture of standing wave consisting of left- & right-moving particles
    - (Sharoglazova *et al.* assume this naive picture is correct in their analysis)

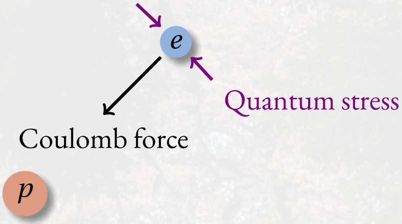
Force breakdown on electron in hydrogen atom

- Polar decomposition—one-particle case:

$$\langle T^{ij}(\boldsymbol{x},t)\rangle_\psi = \mathscr{R}^2\left\{\underbrace{\frac{(\nabla^i\mathscr{S})(\nabla^j\mathscr{S})}{m}}_{\textbf{Particle motion}} + \underbrace{\frac{\hbar^2}{2m\mathscr{R}^2}\Big((\nabla^i\mathscr{R})(\nabla^j\mathscr{R}) - \mathscr{R}(\nabla^i\nabla^j\mathscr{R})\Big)}_{\textbf{Quantum stress}}\right\}$$

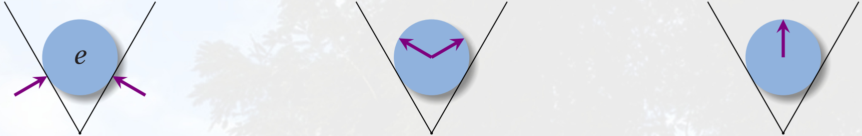
- For static system (ground state), **stresses are forces exerted by pilot wave**.





How do these forces balance?

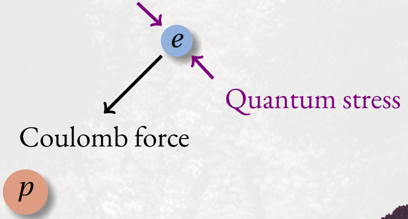
- Tangential forces can sum to radial force for finite-sized object.



  - Imagine electron as finite ball, then take the zero radius limit.
  - Net radial force survives zero-radius limit.

- Net force comes from *divergence* of stress tensor.

$$\mathscr{R}^2 F_Q^j = \mathscr{R}^2(-\nabla^j Q) = -\nabla_i \langle T_Q^{ij} \rangle_\psi$$



- So tangential stress can give radial force:

$$\nabla_i \left[\hat{\theta}^i \hat{\theta}^j + \hat{\phi}^i \hat{\phi}^j \right] = -2 \frac{\hat{r}^j}{|\boldsymbol{r}|}$$