Study of Mutually Unbiased Bases in a Quantum Theory over Fq:

Hampton Smith [1], Tatsu Takeuchi[2]
Virginia Tech [1] Center for Neutrino Physics at Virginia Tech [2]

VIRGINIA TECH.

Project Purpose:

- Mutually Unbiased Bases (MUBs) represent a key mathematical tool for Quantum Information Science
- The known methods of building MUBs can only produce objects in dimension N=qk, q being a prime, and k a positive integer. This suggests a connection to a different set of mathematical tools called the Finite Fields.

- This study seeks to exploit this suspected connection to construct a Quantum Theory (QT) over the Finite Fields (F_q) to study MUBs.
- The algebra of this new QT suggests a mapping to traditional Quantum Mechanics (QM) and as a result a solution to MUBS in arbitrary dimension could be found.

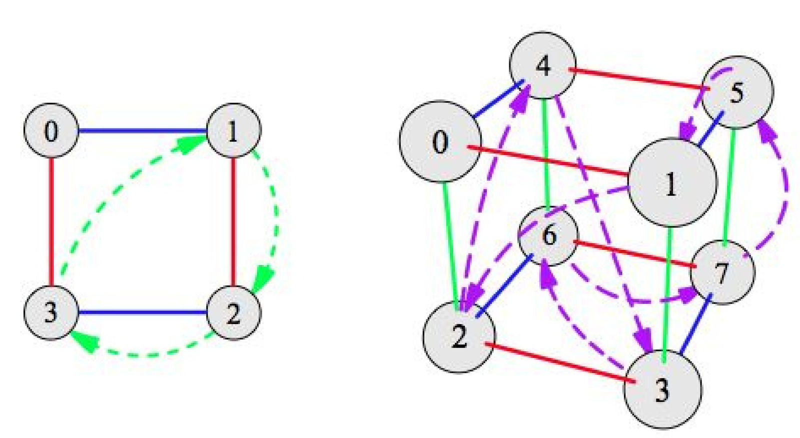

Figure 1: A visualization of addition and multiplication in $\mathbb{F}$. Image Credit: Vladimir Arnold, *Dynamics, Statistics and Projective Geometry of Galois Fields.*

Research Plan:

- We expect to create a computational tool in Mathematica capable of evaluating the structure of a generated QT given any Fq.
- If successful, this tool will then be investigated for generalization to arbitrary dimension.
- Once completed, the Mathematica package will be disseminated to the QIS community.

Background Information:

- The question of MUBs exists at an atypical intersection of surprisingly many scientific fields. The earliest discussion of MUBs can be found in a 1960 paper by Schwinger, [2] and their applications by Ivonovic in 1981.[3]
- A flurry of research into the nature has been performed on MUBs, resulting in the discovery of a surprising number of intersections.
- A related problem, Symmetric Informationally Complete, Positive Operator Value Measures or SIC-POVMs, are computationally neat objects with wide applications in QIS, such as cryptography.The problem of SIC-POVMs happens to be dual to MUBs. MUBs form finite affine planes whereas SICs form finite projective planes.
- A SIC does exist in N = 6 but no clear connection to MUBs can seem to be established with it. [4]
- A connection between SICs and Hilbert's 12th problem was established in 2017.[5]
- As a result of their easily measurable qualities, SICs are being heavily considered as a sort of "standard measurement" for use in foundational QM research.[1]
- The main problem we have with constructing this theory is the lack of consistent notation and the different conventions used to manipulate finite fields in their groups. The groups known as Classical Groups or "Groups of Lie Type" have more subtle structure than their cousins over ℝ and ℂ. So the needed objects to conduct computations aren't as clear.

Figure 2: The probability law first posited by Julian Schwinger in his 1960 paper

A vector space V^N, is said to possess Mutually Unbiased Bases (MUBs) if and only if a pair of arbitrary bases $|a_n\rangle$, $|b_m\rangle$, in V^N meet the following criterion:

$$|\langle a_n|b_m\rangle|^2 = \frac{1}{N}, \quad N = \mathrm{Dim}(V^N).$$ [1]

Figure 3: Some of the sample calculations of A field characteristic 5 over a 2nd degree field extension.

3.3 $\mathbb{F}_5^2$

We write the five elements of $\mathbb{F}_5$ as $\{\underline{0},\underline{1},\underline{2},-\underline{2},-\underline{1}\}$ instead of $\{\underline{0},\underline{1},\underline{2},\underline{3},\underline{4}\}$. The addition and multiplication tables of $\mathbb{F}_5 = \{\underline{0},\underline{1},\underline{2},-\underline{2},-\underline{1}\}$ are

+	0	1	2	−2	−1
0	0	1	2	−2	−1
1	1	2	−2	−1	0
2	2	−2	−1	0	1
−2	−2	−1	0	1	2
−1	−1	0	1	2	−2

×	0	1	2	−2	−1
0	0	0	0	0	0
1	0	1	2	−2	−1
2	0	2	−1	1	−2
−2	0	−2	1	−1	2
−1	0	−1	−2	2	1

The bra-kets are:

	$\|a\rangle$	$\|b\rangle$	$\|c\rangle$	$\|d\rangle$	$\|e\rangle$	$\|f\rangle$
$\langle a\|$	1	0	1	1	1	1
$\langle b\|$	0	1	1	−1	2	−2
$\langle c\|$	1	1	2	0	−2	−1
$\langle d\|$	1	−1	0	2	−1	−2
$\langle e\|$	1	−2	−1	−2	2	0
$\langle f\|$	1	2	−2	−1	0	2

Expected Results:

- We expect to use the Mathematica package to establish a mapping between our bespoke QT and traditional quantum mechanics.

- This mapping will consist of a connection between the infinite-dimensional vectors present within the traditional space of QM and the finite vectors present within the toy model.
- Once established, this package will be used to evaluate previous findings on MUBs in traditional QM. We expect to either have matching MUBs or present an alternative structure.

```
genFV[p_, d_] := Table[IntegerDigits[n, p, d], {n, 1, p^d - 1}] (*Generates all members of the finite vec
genNEFV[p_, d_] := Flatten[Table[(PadLeft[Prepend[#, 1], d] & /@ Tuples[Range[0, p - 1], i]), {i, 0, d - 1}
SesquilinearForm[a_, b_, p_] := Mod[Total[(If[# == 0, 0, ModularInverse[#, p]] & /@ a) * b], p, -p / 2] (*Gen
InnerProductTable[p_, d_] := Grid[Table[SesquilinearForm[i, j, p], {i, genNEFV[p, d]}, {j, genNEFV[p, d]
FiniteNorm[a_, b_, p_, d_] := Mod[SesquilinearForm[a, b, p] * SesquilinearForm[b, a, p], p, -p / 2] / Sum[Mo
nonzerovectorfinder[p_, d_] := DeleteCases[genNEFV[p, d], x_ /; Count[x, 0] ≠ 0]
InnerProductNonZeroTable[p_, d_] := Table[SesquilinearForm[i, j, p], {i, nonzerovectorfinder[p, d]}, {j,
Findorthogonalvectors[a_, candidates_, p_, d_] := DeleteCases[candidates, i_ /; Mod[SesquilinearForm[a,
FindvectorBases[a_, candidates_, p_, d_] := NestList[Findorthogonalvectors[a, candidates, p, d], Findort
```

Figure 4: A snapshot of the Mathematica program currently in development. We are in the process of finalizing the implementation of the toy Quantum Theory.

Sources:
[1] https://arxiv.org/pdf/1004.3348.pdf
[2] https://www.jstor.org/stable/70873
[3]https://iopscience.iop.org/article/10.1088/030
5-4470/14/12/019/pdf
[4] https://arxiv.org/pdf/quant-ph/0406032.pdf
[5] https://arxiv.org/pdf/1701.05200.pdf

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