# SPEKKENS' TOY MODEL AND GALOIS FIELD QUANTUM MECHANICS OVER $F_5$

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# Acknowledgments:

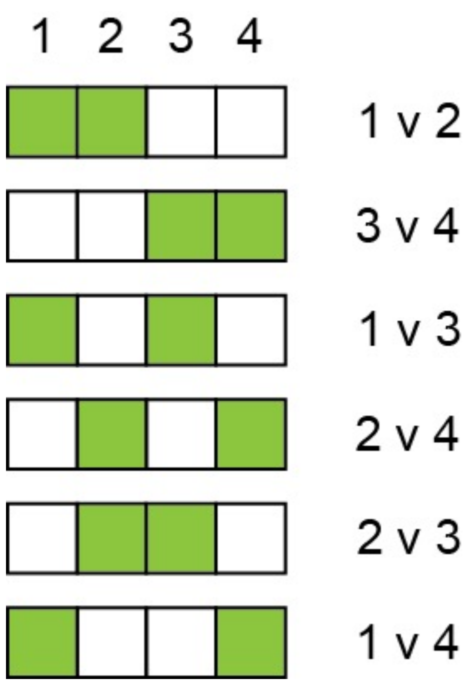
- This talk is based on some work I did in 2018-2019 in collaboration with Djordje and Lay Nam.
- We were starting discussions on a longer paper in 2020 when Lay Nam left us.
- The problem was left on the back burner until now.
- Hope to finish this soon. I welcome new collaborators.

# Spekkens' Toy Model

- Robert W. Spekkens
  "Evidence for the epistemic view of quantum states: A toy theory"
  Phys. Rev. A 75, 032110 (2007)
- Question: are quantum states ontic (represent reality itself) or epistemic (represent our (incomplete) knowledge about reality)?
- Spekkens' toy model assumes the existence of an ontic reality underlying the epistemic "quantum" states
- "Knowledge Balance Principle" :
  If the ontic state of a system requires 2N bits of information to specify, the maximum knowledge one can have about the system at any time is N bits.

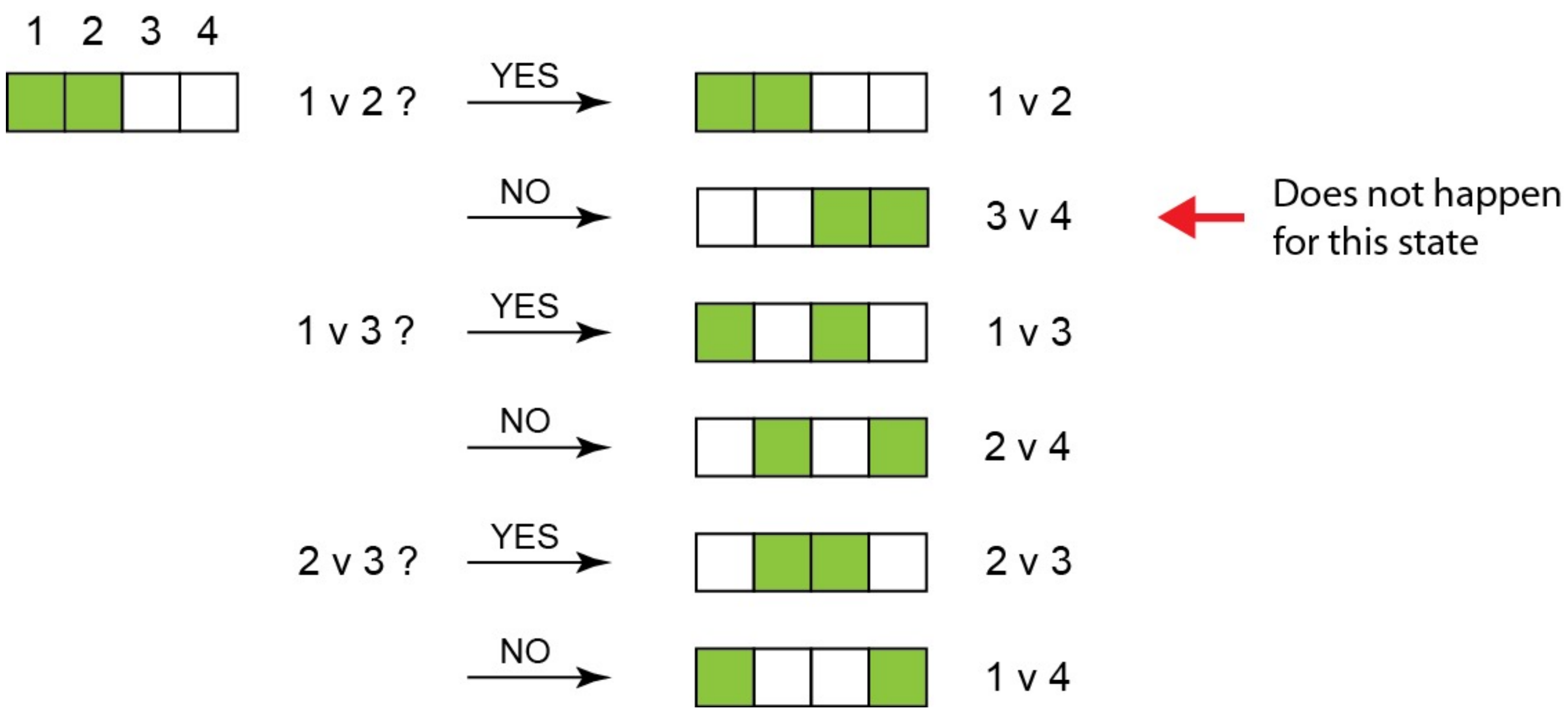
# Spekkens' Elementary System

- 2 bit system = 4 ontic states
- Maximum knowledge about the system can only be 1 bit
  → 6 possible epistemic states

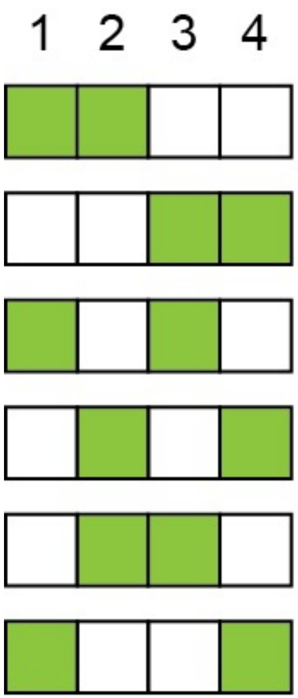


# Measurement in Spekkens' Model

- A "measurement" may or may not knock the ontic state into a new state
- The epistemic state "collapses" onto the "measurement" outcome



# Correspondence to Spin ½ States




1 v 2 $\longleftrightarrow$ $|0\rangle$

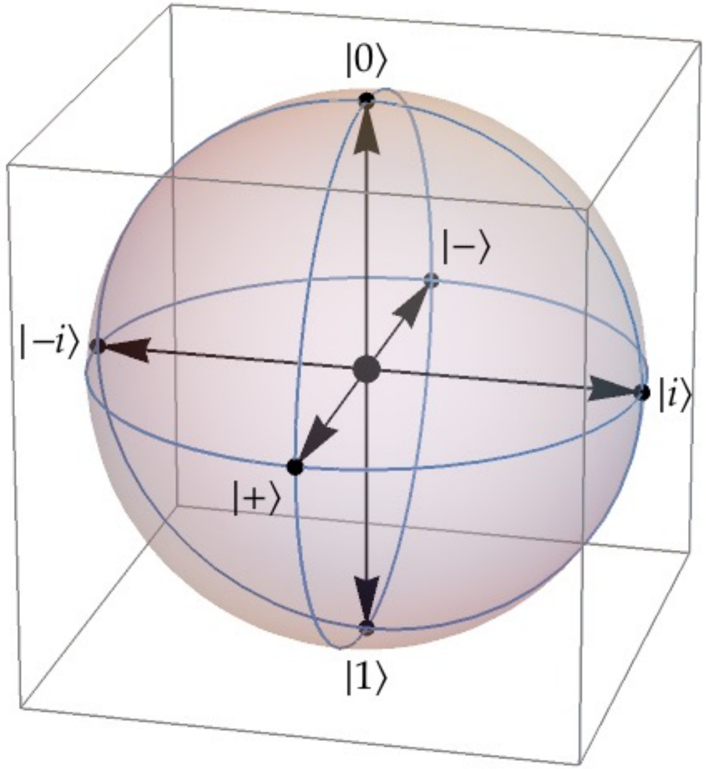
3 v 4 $\longleftrightarrow$ $|1\rangle$

1 v 3 $\longleftrightarrow$ $|+\rangle$

2 v 4 $\longleftrightarrow$ $|-\rangle$

2 v 3 $\longleftrightarrow$ $|+i\rangle$

1 v 4 $\longleftrightarrow$ $|-i\rangle$



# Coherent Superpositions

- Superpositions (FOIL sums) are defined between epistemic states with disjoint ontic support:

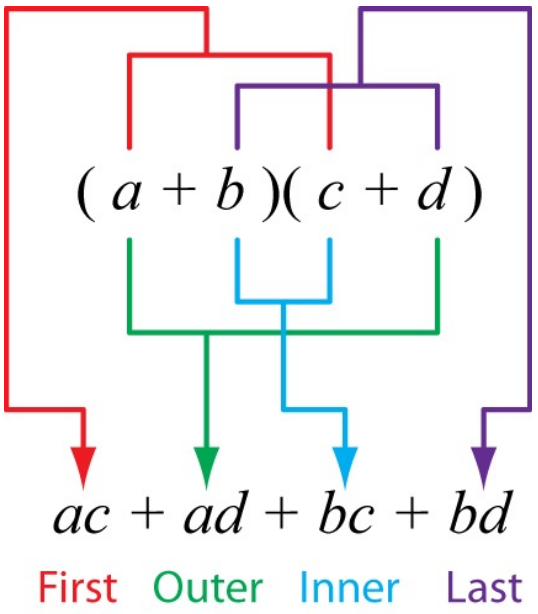
$$(a \vee b) +_1 (c \vee d) = a \vee c \qquad \text{first}$$

$$(a \vee b) +_2 (c \vee d) = b \vee d \qquad \text{last}$$

$$(a \vee b) +_3 (c \vee d) = b \vee c \qquad \text{inner}$$

$$(a \vee b) +_4 (c \vee d) = a \vee d \qquad \text{outer}$$

$a, b, c, d$ are all distinct



# Example 1 ☺

- FOIL sums of (1v2) and (3v4) :

$$(1 \vee 2) +_1 (3 \vee 4) = 1 \vee 3 \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$$

$$(1 \vee 2) +_2 (3 \vee 4) = 2 \vee 4 \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$$

$$(1 \vee 2) +_3 (3 \vee 4) = 2 \vee 3 \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = |+i\rangle$$

$$(1 \vee 2) +_4 (3 \vee 4) = 1 \vee 4 \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) = |-i\rangle$$

# Example 2 ☺

- FOIL sums of (2v3) and (1v4) :

$$(2\text{V}3) +_1 (1 \vee 4) = (2 \vee 1) \quad \leftrightarrow \quad \frac{1}{\sqrt{2}} (|{+i}\rangle + |{-i}\rangle) = |0\rangle$$

$$(2\text{V}3) +_2 (1 \vee 4) = (3 \vee 4) \quad \leftrightarrow \quad \frac{1}{\sqrt{2}} (|{+i}\rangle - |{-i}\rangle) = |1\rangle$$

$$(2\text{V}3) +_3 (1 \vee 4) = (3 \vee 1) \quad \leftrightarrow \quad \frac{1}{\sqrt{2}} (|{+i}\rangle + i|{-i}\rangle) = e^{\frac{i\pi}{4}}|+\rangle$$

$$(2\text{V}3) +_4 (1 \vee 4) = (2 \vee 4) \quad \leftrightarrow \quad \frac{1}{\sqrt{2}} (|{+i}\rangle - i|{-i}\rangle) = e^{-\frac{i\pi}{4}}|-\rangle$$

# Example 3 ☹

- FOIL sums of (1v3) and (2v4) :

$$(1 \vee 3) +_1 (2 \vee 4) = (1 \vee 2) \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) = |0\rangle$$

$$(1 \vee 3) +_2 (2 \vee 4) = (3 \vee 4) \quad \leftrightarrow \quad \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = |1\rangle$$

$$(1 \vee 3) +_3 (2 \vee 4) = (3 \vee 2) \quad \nleftrightarrow \quad \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle) = e^{\frac{i\pi}{4}}|-i\rangle$$

$$(1 \vee 3) +_4 (2 \vee 4) = (1 \vee 4) \quad \nleftrightarrow \quad \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle) = e^{-\frac{i\pi}{4}}|+i\rangle$$

- Changing the ordering of the ontic state labels will also change the sums ☹

# Problems:

1. Sums depend on the ordering of the ontic labels of the epistemic states (flipping of the labels cannot be identified with phase change)

2. Superposition of states with un-disjoint ontic support are not defined

3. Without a vector space (sums of arbitrary pairs of vectors are well-defined), how can multi-particle states be defined without anything to tensor?

# Can we make the model more "quantum?"

- Can Spekkens' toy model be mapped onto a linear vector space in which the superposition of arbitrary states is well defined?

- Yes, use a vector space over Galois Fields!
  - L. N. Chang, Z. Lewis, D. Minic, and T. Takeuchi
    "Galois Field Quantum Mechanics"
    Mod. Phys. Lett. B 27 (2013) 1350064
  - L. N. Chang, Z. Lewis, D. Minic, and T. Takeuchi
    "Spin and Rotations in Galois Field Quantum Mechanics"
    Journal of Physics A: Math. Theor. 46 (2013) 065304
  - L. N. Chang, Z. Lewis, D. Minic, and T. Takeuchi
    "Biorthogonal Quantum Mechanics: Super-Quantum Correlations and Expectation Values without Definite Probabilities"
    Journal of Physics A: Math. Theor. 46 (2013) 485306
  - L. N. Chang, Z. Lewis, D. Minic, and T. Takeuchi
    "Quantum $F_{un}$: The q=1 Limit of Galois Field Quantum Mechanics, Projective Geometry, and the Field with One Element"
    Journal of Physics A: Math. Theor. 47 (2014) 405304

# What are Galois Fields?

- Examples:

$$F_2 = \{\underline{0}, \underline{1}\}$$

$$F_3 = \{\underline{0}, \underline{1}, \underline{2}\}$$

$$F_4 = F_2[\omega] = \{\underline{0}, \underline{1}, \underline{\omega}, \underline{\omega}^2\}, \quad \underline{1} + \underline{\omega} + \underline{\omega}^2 = \underline{0}$$

$$F_5 = \{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}\}$$

$$F_7 = \{\underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4}, \underline{5}, \underline{6}\}$$

$$F_8 = F_2[\varepsilon] = \{\underline{0}, \underline{1}, \underline{\varepsilon}, \underline{1} + \underline{\varepsilon}, \underline{\varepsilon}^2, \underline{1} + \underline{\varepsilon}^2, \underline{\varepsilon} + \underline{\varepsilon}^2, \underline{1} + \underline{\varepsilon} + \underline{\varepsilon}^2\}, \quad \underline{1} + \underline{\varepsilon} + \underline{\varepsilon}^3 = \underline{0}$$

$$F_9 = F_3[i] = \{\underline{0}, \underline{1}, \underline{2}, \underline{i}, \underline{2i}, \underline{1} + \underline{i}, \underline{1} + \underline{2i}, \underline{2} + \underline{i}, \underline{2} + \underline{2i}\}, \quad \underline{1} + \underline{i}^2 = \underline{0}$$

- $F_q$ where $q = p^n$, $p$ prime, $n \in \mathbb{N}$
- Addition and multiplication defined modulo $p$

# The Galois Field $F_5$

- $F_5 = \{\, \underline{0}, \underline{1}, \underline{2}, \underline{3}, \underline{4} \,\} = \{\, \underline{0}, \underline{1}, \underline{2}, -\underline{2}, -\underline{1} \,\}$

| $+$ | $\underline{0}$ | $\underline{1}$ | $\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ |
|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{0}$ | $\underline{1}$ | $\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ |
| $\underline{1}$ | $\underline{1}$ | $\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ | $\underline{0}$ |
| $\underline{2}$ | $\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ | $\underline{0}$ | $\underline{1}$ |
| $-\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{2}$ |
| $-\underline{1}$ | $-\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{2}$ | $-\underline{2}$ |

| $\times$ | $\underline{0}$ | $\underline{1}$ | $\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ |
|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ | $\underline{0}$ |
| $\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{2}$ | $-\underline{2}$ | $-\underline{1}$ |
| $\underline{2}$ | $\underline{0}$ | $\underline{2}$ | $-\underline{1}$ | $\underline{1}$ | $-\underline{2}$ |
| $-\underline{2}$ | $\underline{0}$ | $-\underline{2}$ | $\underline{1}$ | $-\underline{1}$ | $\underline{2}$ |
| $-\underline{1}$ | $\underline{0}$ | $-\underline{1}$ | $-\underline{2}$ | $\underline{2}$ | $\underline{1}$ |

- Note that $(\pm\underline{2})^2 = -\underline{1} \;\rightarrow\; \pm\underline{2}$ correspond to $\pm i$
- No square root of 2.

# 2D vector space over $F_5$

- Six non-zero inequivalent states:

$$|A\rangle = \begin{bmatrix} \underline{1} \\ \underline{0} \end{bmatrix}, \quad |C\rangle = \begin{bmatrix} \underline{1} \\ \underline{1} \end{bmatrix}, \quad |E\rangle = \begin{bmatrix} \underline{1} \\ \underline{2} \end{bmatrix}$$

$$|B\rangle = \begin{bmatrix} \underline{0} \\ \underline{1} \end{bmatrix}, \quad |D\rangle = \begin{bmatrix} \underline{1} \\ -\underline{1} \end{bmatrix}, \quad |F\rangle = \begin{bmatrix} \underline{1} \\ -\underline{2} \end{bmatrix}$$

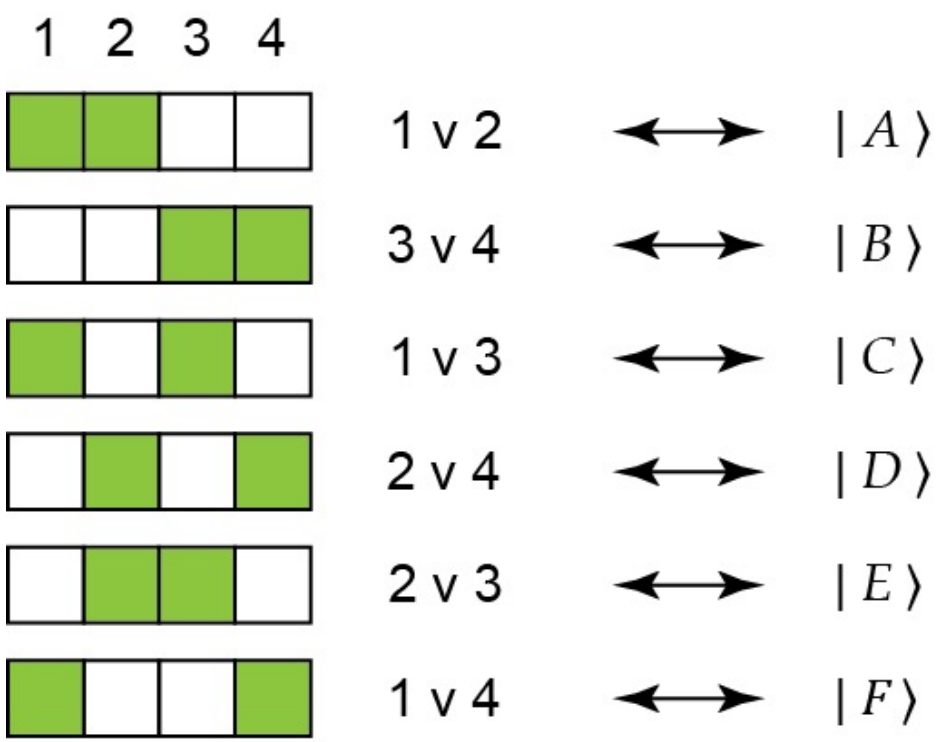
- cf.

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |+i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix},$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad |-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

# Correspondence to Spekkens

- Spekkens' elementary system can be endowed with a full linear structure via the correspondence



# Superpostions in Spekkens' model

- In addition to

$$(1\vee 2)+_1(3\vee 4)=(1\vee 3) \quad \leftrightarrow \quad |A\rangle+|B\rangle=|C\rangle$$

$$(1\vee 2)+_2(3\vee 4)=(2\vee 4) \quad \leftrightarrow \quad |A\rangle-|B\rangle=|D\rangle$$

$$(1\vee 2)+_3(3\vee 4)=(2\vee 3) \quad \leftrightarrow \quad |A\rangle+\underline{\underline{2}}|B\rangle=|E\rangle$$

$$(1\vee 2)+_4(3\vee 4)=(1\vee 4) \quad \leftrightarrow \quad |A\rangle-\underline{\underline{2}}|B\rangle=|F\rangle$$

we have

$$(1\vee 2)+_1(1\vee 3)=(1\vee 4) \quad \leftrightarrow \quad |A\rangle+|C\rangle=\underline{\underline{2}}|F\rangle$$

$$(1\vee 2)+_2(1\vee 3)=(3\vee 4) \quad \leftrightarrow \quad |A\rangle-|C\rangle=-|B\rangle$$

$$(1\vee 2)+_3(1\vee 3)=(2\vee 4) \quad \leftrightarrow \quad |A\rangle+\underline{\underline{2}}|C\rangle=-\underline{\underline{2}}|D\rangle$$

$$(1\vee 2)+_4(1\vee 3)=(2\vee 3) \quad \leftrightarrow \quad |A\rangle-\underline{\underline{2}}|C\rangle=-|E\rangle$$

etc.

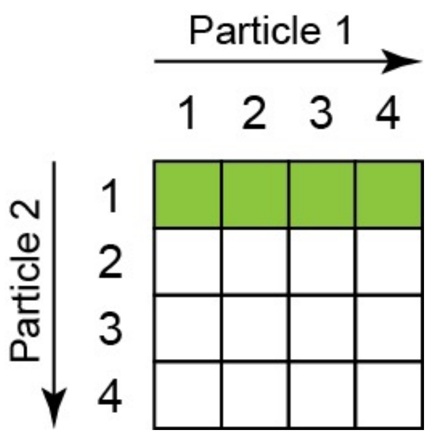
# Multiparticle states and Entanglement

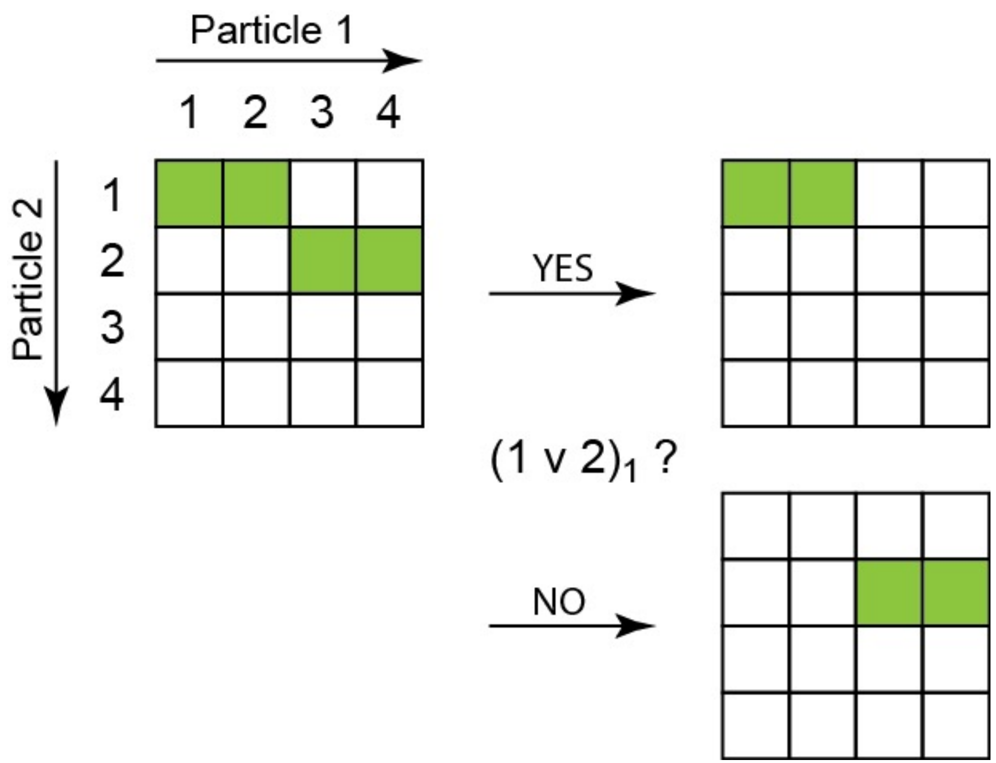
- A vector space description of the system will allow us to construct multiparticle states via tensoring
- $F_5^2 \otimes F_5^2 = F_5^4$ has $(5^4 - 1)/4 = 156$ inequivalent states of which $6^2 = 36$ are product states and $156 - 36 = 120$ are entangled states
- How are multiparticle states and entanglement treated in Spekkens' model?

# 2-particle states in Spekkens' model

- A pair of elementary systems in Spekkens' model will have $2^4=16$ ontic states
- The Knowledge Balance Principle demands not only that 2 bits of information is all that can be known about the combined system, but that only 1 bit of information can be known for each subsystem
- This principle must be maintained after successive measurements

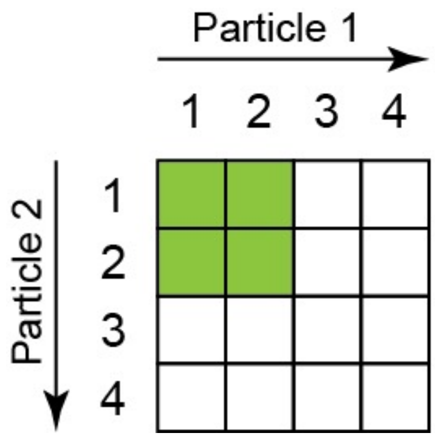
# Examples of Disallowed States



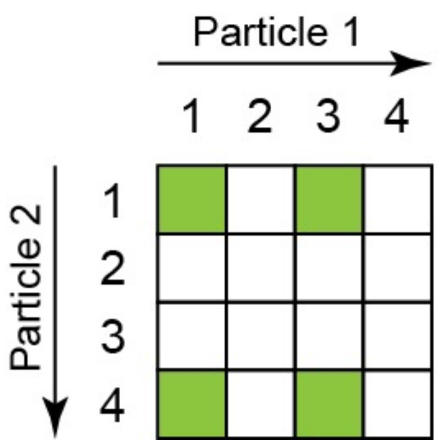



# Allowed States 1 : Product States

- There are $6^2 = 36$ product states, e.g.



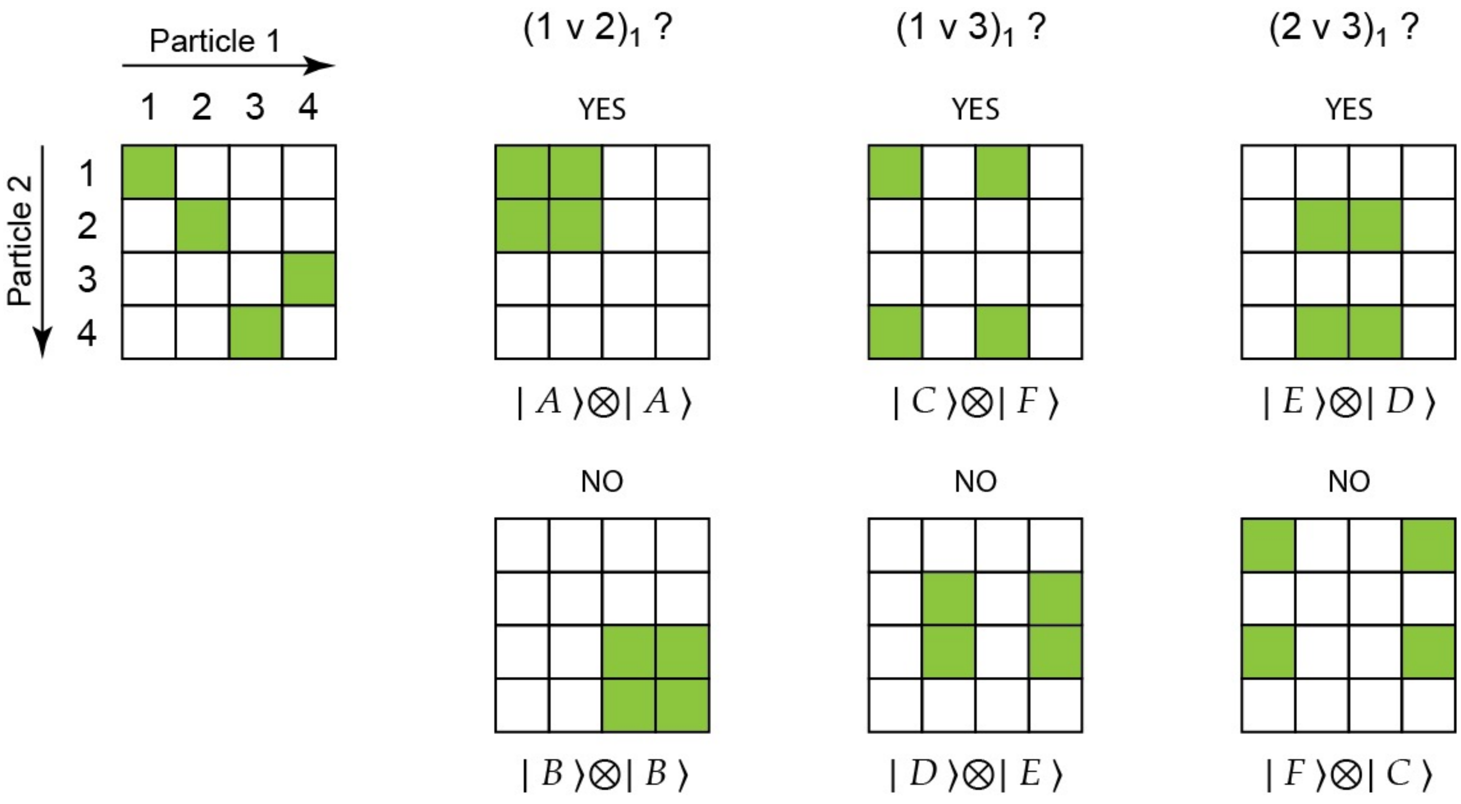

$(1 \text{ v } 2)\cdot(1 \text{ v } 2) \longleftrightarrow |A\rangle \otimes |A\rangle$




$(1 \text{ v } 3)\cdot(1 \text{ v } 4) \longleftrightarrow |C\rangle \otimes |F\rangle$

# Allowed States 2 : Entangled States

- There are 4! = 24 entangled states, e.g.




$$2|A\rangle \otimes |A\rangle + |B\rangle \otimes |B\rangle = |C\rangle \otimes |F\rangle + |D\rangle \otimes |E\rangle = |E\rangle \otimes |D\rangle + |F\rangle \otimes |C\rangle$$

# Mismatch of Entangled States

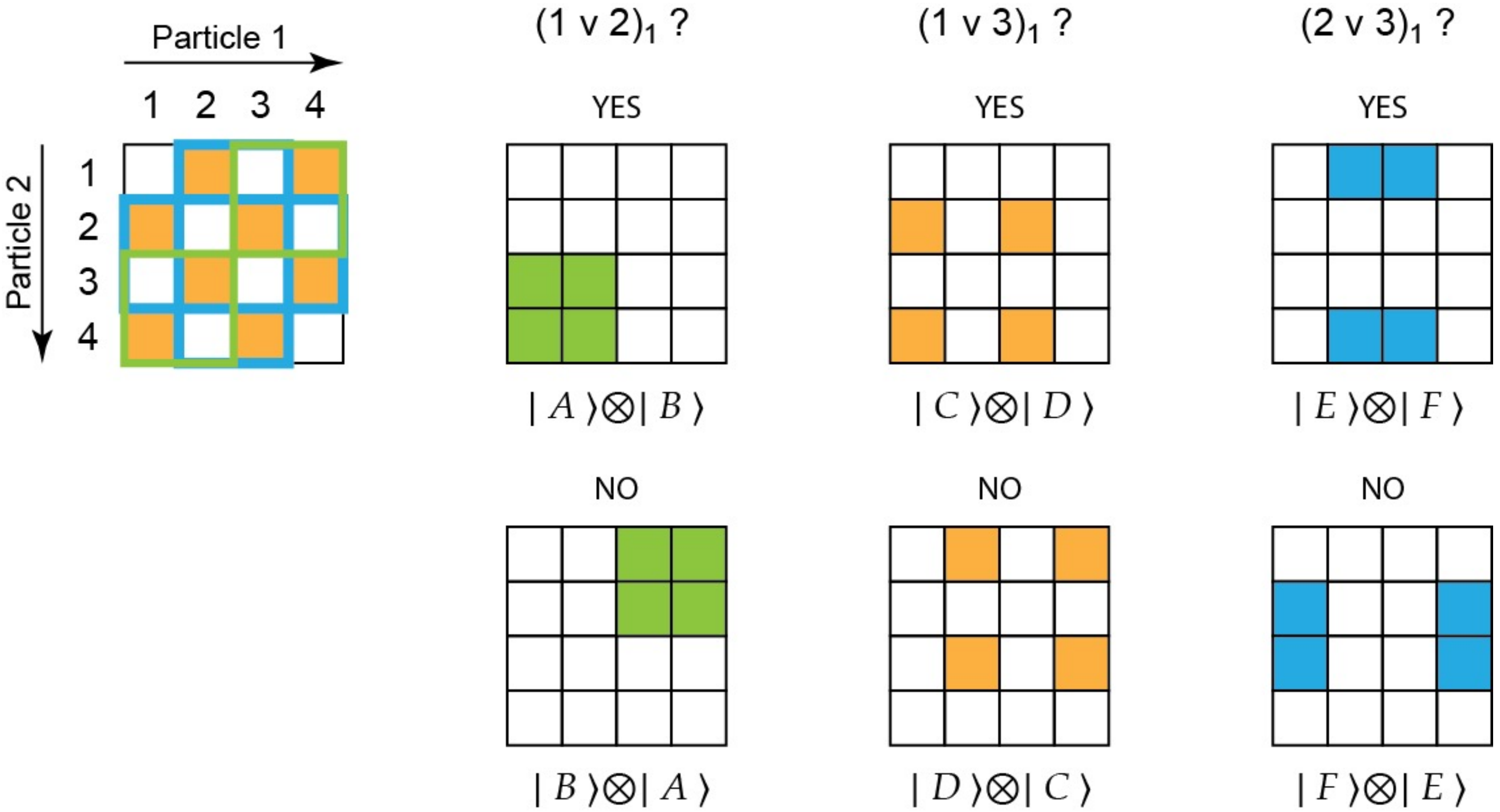
- There are 24 entangled states in Spekkens' model, but 120 entangled states in the tensored 2-particle vector space over $F_5$

- Questions:
  1. Which tensored states are missing?
  2. Can all Spekkens states described by tensored states?

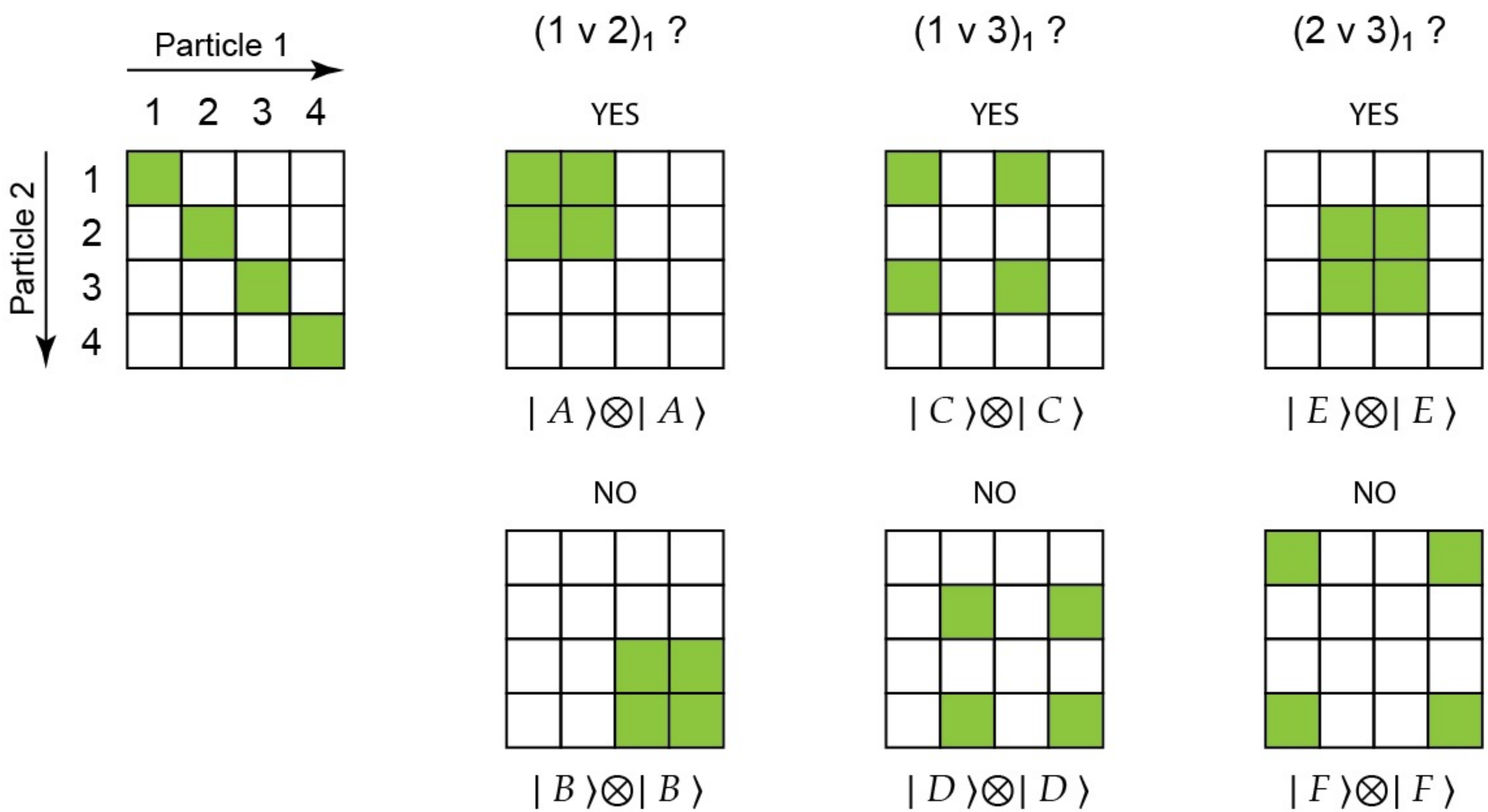
# Tensored State without a Spekkens State

- "Spin" singlet state

$$|A\rangle \otimes |B\rangle - |B\rangle \otimes |A\rangle = -2\left(|C\rangle \otimes |D\rangle - |D\rangle \otimes |C\rangle\right) = |E\rangle \otimes |F\rangle - |F\rangle \otimes |E\rangle$$



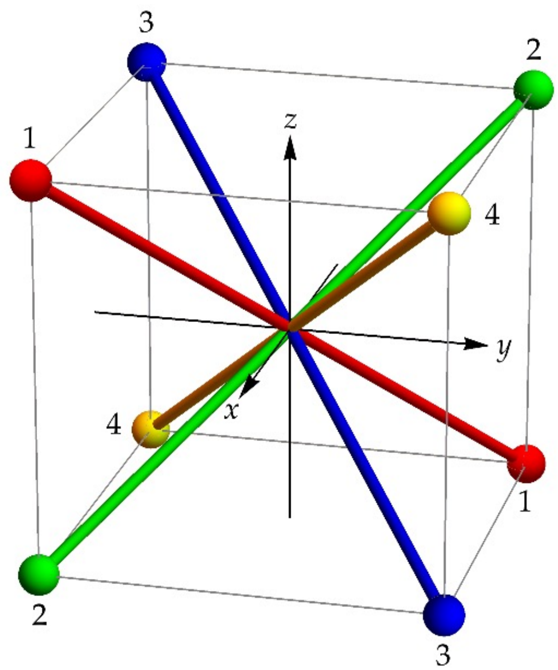
# Spekkens State without a Tensored State

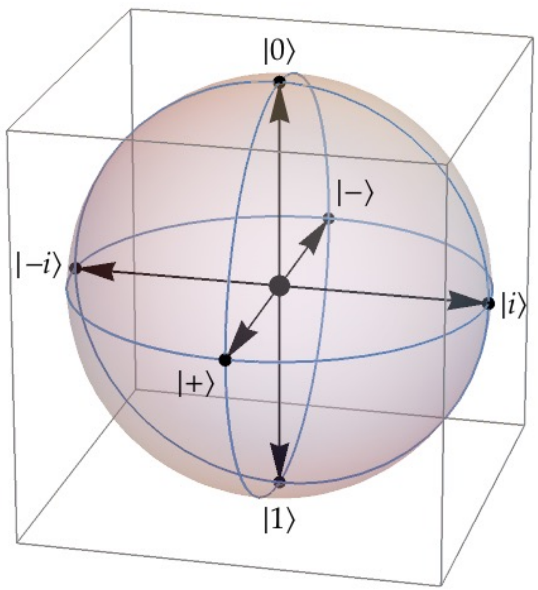



$$|A\rangle \otimes |A\rangle + |B\rangle \otimes |B\rangle = -2(|C\rangle \otimes |C\rangle + |D\rangle \otimes |D\rangle) = 2(|E\rangle \otimes |F\rangle + |F\rangle \otimes |E\rangle)$$

# Symmetries of Spekkens' States and Spins

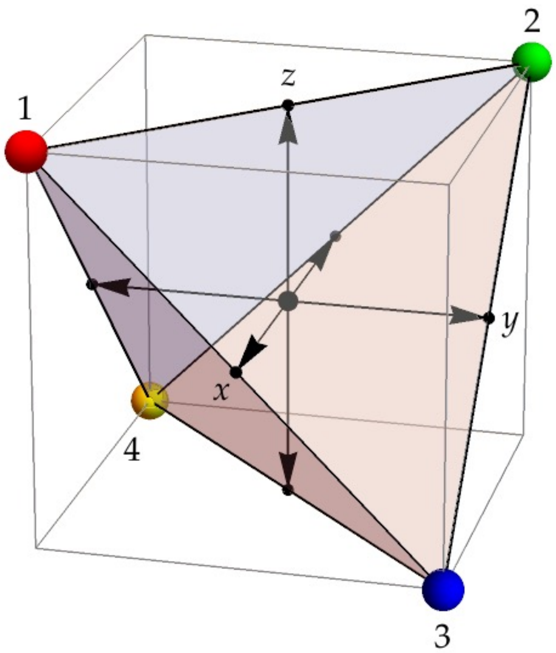
- Spekkens' elementary system has an $S_4$ symmetry
- $S_4$ has 2-dimensional irreducible projective representations over $C$ and $F_5$, the former of which can be mapped to the spin-1/2 representation of $SO(3)$

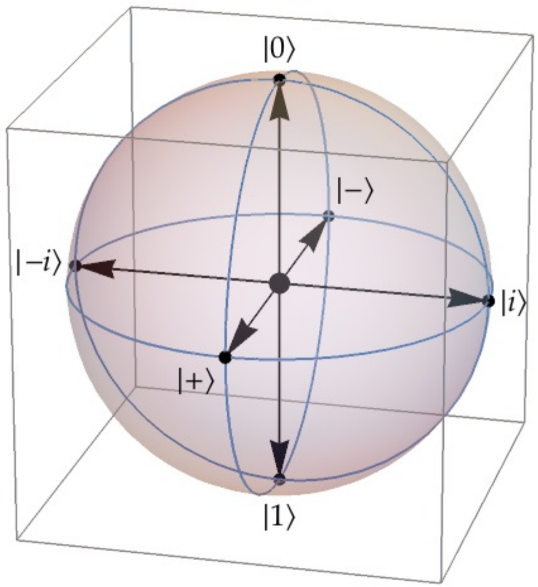


# But the Mapping is Chiral!

- There is a mismatch in the symmetries of Spekkens' toy model and "spins" in canonical and $F_5$ quantum mechanics:
  - Even permutations → Rotations
  - Odd permutations → Reflections → Complex Conjugation?

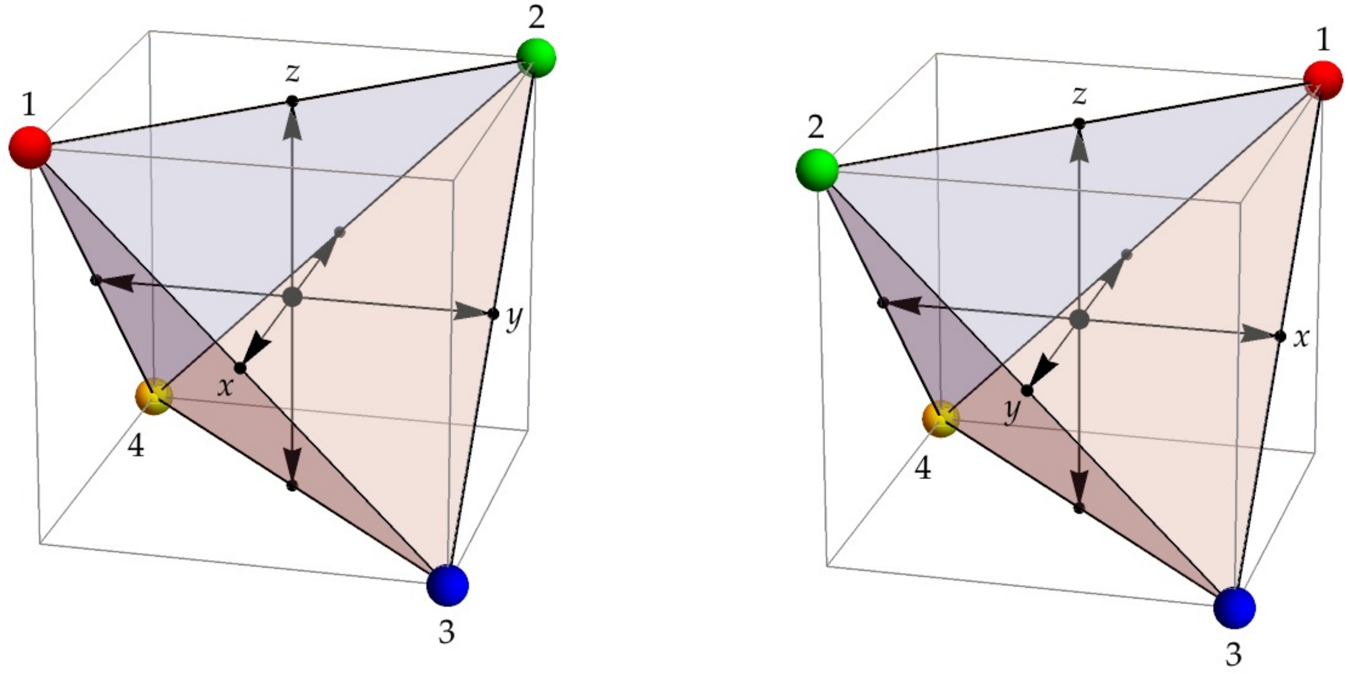



*This is also the reason why the FOIL sums are ill defined.

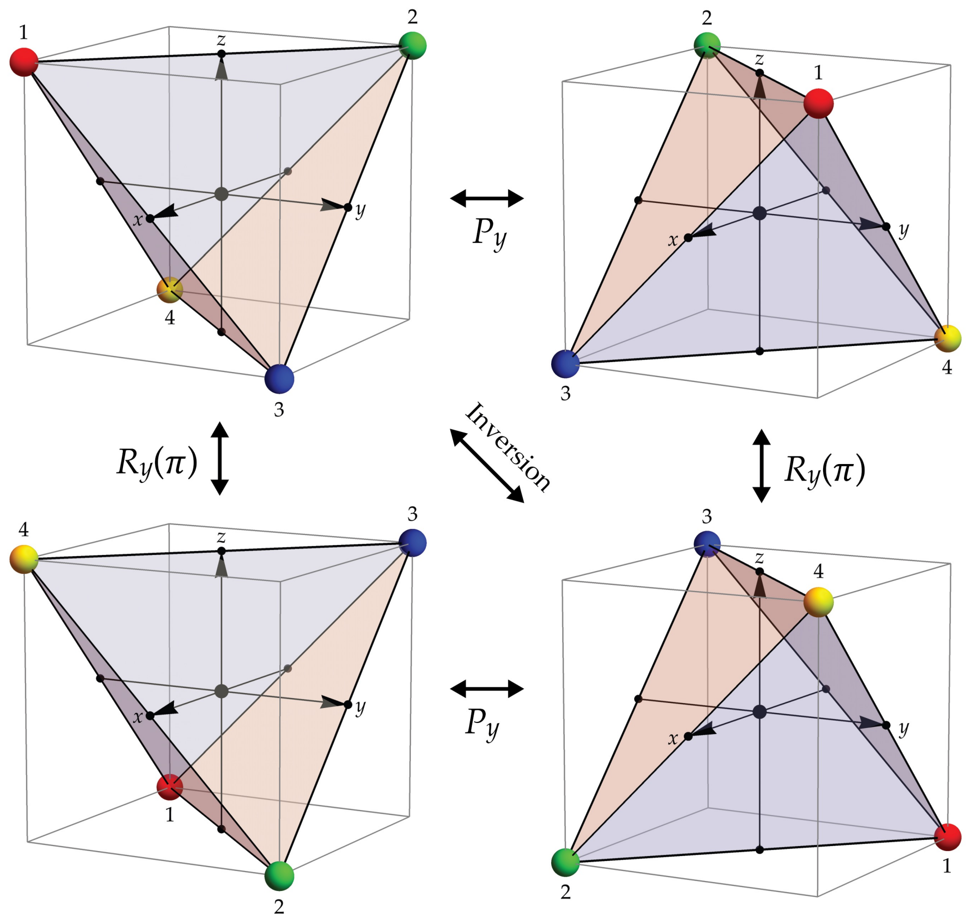
# Majorana?

- In Spekkens' toy model (1v2) and (2v1) are supposed to be the same epistemic state
- But a (12) permutation maps the corresponding "spin" state to the opposite chirality.  How can these be the "same" state?  Majorana?



# 4D Representation?

- $S_4$ also has 4-dimensional irreducible projective representations over $C$ and $F_5$, the former of which can be mapped to the spin-1/2 representation of $SO(3,1)$



# Work in Progress

- Full characterization of all Spekkens and tensored entangled states
  - Clarify the demands of linearity

- Search for a vector space which better represents Spekkens' model
  - Mapping of Spekkens' model onto Majorana Fermions in canonical and $F_5$ quantum mechanics
  - May require modifications to Spekkens' model, e.g. introduction of chirality (hints of a new type of "quantum logic?")