Optimal shot-put release angle revisited: Familiar and novel perspectives

Michael J. Ruiz[1] and R.K.P. Zia[2]

[1] *Department of Physics & Astronomy, University of North Carolina at Asheville, Asheville, NC 28804*

[2] *Center for Soft Matter and Biological Physics, Department of Physics, Virginia Polytechnic Institute & State University, Blacksburg, VA 24061*

Department of Physics, University of Houston, Houston, TX,

(Dated: August, 2022)

Abstract

It is well known that 45° is the optimal angle to send a projectile to achieve the maximal range, R (on flat ground). Yet, the best shot putters release at angles about 10% lower instead. The resolution to this "puzzle" was traced to the landing site being lower than the height of release. Instead of solving this optimization problem using standard calculus, we revisit alternatives relying on no calculus but only geometry. We highlight the fact that, in the optimal trajectory, the initial and final velocities are always orthogonal and the optimal angle follows easily from this condition and energy conservation. Novel perspectives include (a) the notion of "duality" between the launch and landing and (b) the envelope of all trajectories launched from one point (say the origin of a coordinate system) with the same speed. Consequences associated with these ideas are explored.

I. INTRODUCTION

Although the control of projectile motion had been undoubtedly a major concern of humans for millions of years, quantitative analysis of the problem started just a few centuries ago. The simple version of it – motion under the influence of uniform gravitational acceleration g – is present in all elementary physics texts. Discussed by many publications, the problem of optimizing the effort has been posed in a large variety of situations. A standard one is: Given the initial speed (s, "muzzle velocity"), what angle (α, relative to the horizontal) should the projectile be launched for it to reach a maximum distance ("range" ρ). Of course, if the landing site is at the same height (elevation) as the launch site, it's well known that $\alpha_{opt} = 45°$, a result that Galileo found[1]. For the case where the landing site is lower than the launch height, a number of papers were published from 1977 [2–8]. The title of this paper follows that in several of these[3, 5, 7], as the problem of finding α_{opt} is often cast in the light of a "puzzle": Why do the best shot putters (those who achieve the maximal range) release their shot with an average[9] of 37° rather than 45°? The resolution for $\alpha_{opt} < 45°$ turned out to be the shot hitting the ground *below* the height of release, though the issue for the athlete is more complex, involving both physiology and physics. As a concrete example in physics, Lichentenberg and Wills [3] arrived at $\alpha_{opt} \simeq 42.3°$ based on a drop of $2.1m$ and $\rho = 22m$ (and thus, associated with a minimal s). For the same reason, to optimize a basketball shot[10] (for which the landing site is *above* the launch site), we find $\alpha_{opt} > 45°$. Thus, we are motivated to consider the less widely known case of projectile motion, in which the heights of the launch and landing sites differ by a general value, δ. As an optimization problem, the standard route is to find a relationship between $\{s, \delta, \alpha, \rho\}$, set the derivative $d\rho/d\alpha$ to zero (with fixed $\{s, \delta\}$, say) and solve for α_{opt}. Typically this approach involves multiple steps and laborious algebra.[3] A much simpler way requires no calculus; only the concepts of vectors and the cross product,[5] or indeed, only geometry.[7]

In this paper, we revisit the basketball/shot-put problem and present a comprehensive version of the purely geometric solution. With minimal algebra (not even a quadratic equation) and trigonometry (no further than the definition of cotangent of an angle), our solution requires no more than middle school mathematics. As for knowledge in physics, this approach requires only the simplest consequences of $\vec{F} = m\vec{a}$ (constant g) and energy conservation.

First, we relate this problem to an amusing "brain teaser" in geometry. In addition, we end by bringing together different perspectives in several different fields of physics/geometry. In particular, we believe the notion of "duality," as introduced into this problem, is novel. Further, we believe it is valuable for the student to see that subtle connections between different branches of physics exist (geometric optics and projectile motion here; quantum and statistical mechanics elsewhere). The remainder of this article is organized as follows. We begin with a mathematical "appetizer," which will shed light on the problem at hand. The next section sets up the physics behind the shot put question and provides a variety of results. A following section is devoted to two novel perspectives of this venerable problem. We end with some concluding remarks.

II. A SIMPLE "BRAIN TEASER" IN GEOMETRY

If a rectangle is inscribed in a quarter-circle (gray, in Fig. 1a), what aspect ratio will maximize its area? The answer is intuitively clear and easy to show, namely, 1 : 1. If we now extend this question to a rectangle inscribed in two quarter circles with *different radii* in two adjacent quadrants (gray, in Fig. 1b), what aspect ratio is required? To be specific, let us denote the radii by $L_{1,2}$, and the heights and widths of the two parts of the rectangle by $H_{1,2}$ and $W_{1,2}$ respectively, as shown in Fig. 1b. By construction, we require

$$H_1 = H_2 \equiv H, \tag{1}$$

and so, the question is: Given the L values, what are the ratios H/W_i which maximize the total area $A \equiv H\,(W_2 + W_2)$?

The standard approach is to substitute $W_i = \sqrt{L_i^2 - H^2}$ into A, set $dA/dH = 0$, and solve for H. But, can a solution be found *without* using calculus. We show that, indeed, this optimization problem can be solved with geometry alone.

We can rephrase the question to: What angle (γ) between the diagonals (dashed lines) in Fig. 1b maximizes A? Fig. 2 shows a sequence of (self explanatory) moves of triangles demonstrating that the area of the original rectangle in Fig. 2a is the same as the one in Fig. 2e. Meanwhile, the heavy dashed lines in the former are congruent with those in the latter. In particular, the top edge of the last rectangle touches the end of the slanted dashed line, by construction. Thus, it is clear that the area is *maximal* when the angle between the dashed lines are *orthogonal*. The attentive reader will notice that this is merely a geometric way to show that the area of the rectangle in Fig 1a is just $L_1 L_2 \sin\gamma$, which is of course maximal when $\gamma = 90°$. Also clear is that this result is just the vector product relation $\vec{L}_1 \times \vec{L}_2 = \left|\vec{L}_1\right| \left|\vec{L}_2\right| \sin\gamma$, which is also the area of the parallelogram spanned by L_1 and L_2. Indeed, this is the basis behind the solution in an early publication[5].

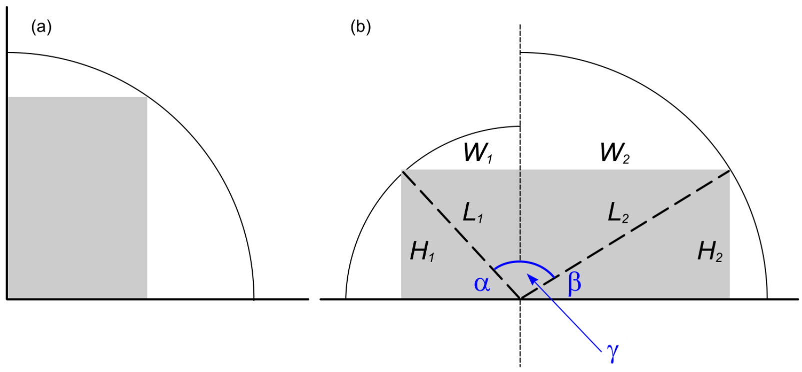

FIG. 1: (a) Rectangle inscribed in a quarter circle. What aspect ratio maximizes the area? (b) Rectangle inscribed in two quarter circles with different radii. What aspect ratio maximizes the area in this case?

So, the conclusion of this brief "brain teaser" is that the area in Fig 1b is maximal when $\gamma = 90°$. Moreover, it is straightforward to note that this property implies that the three rectangles ($H_1 \times W_1$, $W_2 \times H_2$, and $L_1 \times L_2$) are all similar. Thus, we have

$$\frac{H_1}{W_1} = \frac{W_2}{H_2} = \frac{L_2}{L_1}. \tag{2}$$

III. PHYSICS OF PROJECTILE MOTION ON A FLAT EARTH

Since the projectile motion is restricted to a vertical plane, we will denote the horizontal and vertical coordinates by (x, y). For convenience, let the projectile be launched from the origin $(0, 0)$, with velocity $\vec{v} = (v_x, v_y)$ and speed $s = |\vec{v}|$, subjected to a uniform gravitational force, so that its acceleration is the constant $(0, -g)$. One significant length associated with this problem is

$$h_{\max} \equiv s^2/2g, \tag{3}$$

which is the height beyond which this projectile can never reach. Meanwhile, $2h_{\max}$ is the maximum *range* when the project is to land "on flat ground" (at the same height as the

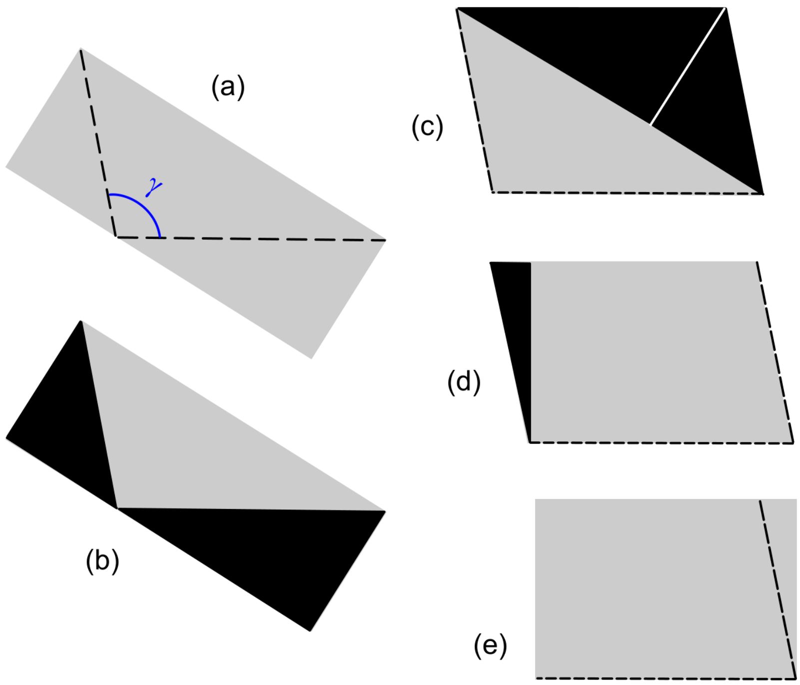

FIG. 2: Sequence of moves demonstrating that the area of the original rectangle (a) is the same as (e).

launch site).

The evolution in time (t) is standard:

$$x(t) = v_x t; \quad y(t) = v_y t - \frac{g}{2} t^2. \tag{4}$$

Eliminating t, we get the equation for the trajectory

$$y(x) = \frac{v_y}{v_x} x - \frac{g}{2v_x^2} x^2 \tag{y(x)}$$

which is the familiar (downward pointing) parabola). While the coefficient of x is clearly unitless, that of x^2 can be written in a form to display its units manifestly, i.e.,

$$y(x) = \frac{v_y}{v_x} x - \frac{1}{2\lambda} x^2, \tag{y(x)Lamb}$$

with

$$\lambda \equiv v_x^2/g. \tag{5}$$

The second derivative $d^2y/dx^2 = -1/\lambda$ is constant for our parabola and λ is the *radius of curvature* at its apex. To associate a physically meaningful length with , we note that, if the projectile were launched at 45°, $v_x^2 = s^2/2$, we would have $\lambda = h_{\max}$ from Eq. (3) and Eq. (5). It is significant that λ does not depend on the y component of $\vec{v}$.

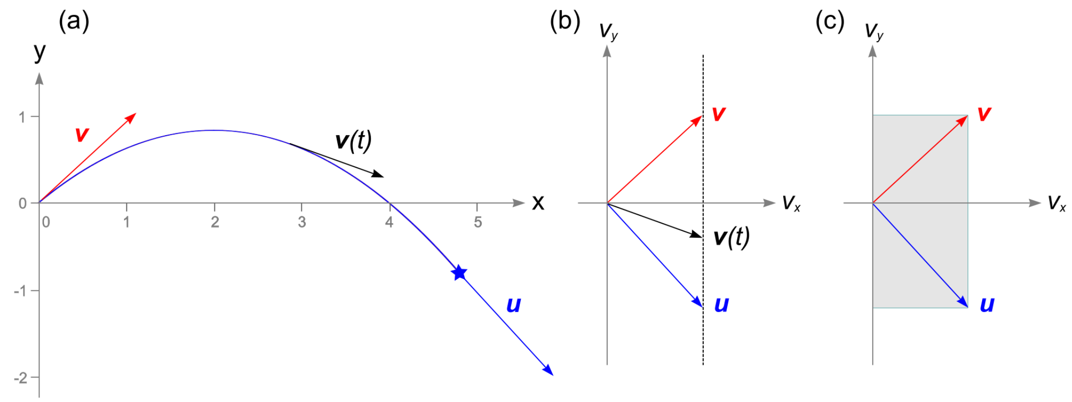

FIG. 3: (a) A projectile is launched at the origin and lands at the blue star. (b) Since v_x is a constant, the tips of the arrows lie on the vertical dashed line.(c) Rectangle highlighting area formed by vectors $\vec{v}$ and $\vec{u}$

The standard way to find the optimal launch angle (α_{opt}) is to write Eqn. (y(x)) as $y\left(x, \alpha|g, s\right)$ and invert it to $x\left(y, \alpha|g, s\right)$. Then we set $dx\left(\delta, \alpha|g, s\right)/d\alpha = 0$ and solve to obtain α_{opt}. Clearly, this is a tedious procedure, prone to many algebraic missteps. A more elegant way is to consider the problem in *velocity* space. In Fig. 3(a), we provide a sketch for a trajectory, as well as the velocities at launch ($\vec{v}$), landing ($\vec{u}$) and a typical location along the way at some time t: $\vec{v}\left(t\right)$. These examples are then plotted in the v_x-v_y plane in Fig 3(b). Since there is no force in the x direction, v_x is a constant of the motion, so that the tips of these arrows all lie on the vertical dashed line. Meanwhile, as the acceleration is a constant in y, we have

$$v_y\left(t\right) = v_y\left(0\right) - gt, \tag{6}$$

so that the tips simply "move" (at uniform "speed") down the dashed line. At landing, occurring at t_{end}, we have $u_y = v_y\left(0\right) - gt_{end}$ so that

$$gt_{end} = v_y\left(0\right) - u_y. \tag{7}$$

Note that, as we are focusing on a projectile on the downward part of the parabolic path, u_y is *negative*. Thus, it is trivial to see that the right side of the above is just the "distance" between the tips of $\vec{v}$ and $\vec{u}$., i.e., the "length" of the gray rectangle in Fig 3(c). Meanwhile, the "width" is precisely v_x so that the area of this rectangle is just $v_x g t_{end}$. But, $v_x t_{end}$ is nothing but the physical distance between the launch and landing sites, i.e., the range ρ. So, we finally arrive at

$$g\rho = \textit{rectangle area.} \tag{8}$$

Comparing Fig.1(b) in the "brain teaser" with Fig. 3(c) here, we see that our problem of maximizing the range (ρ) is precisely the same[22] as maximizing the rectangle's area (A), while the "lengths" $L_{1,2}$ correspond to the magnitudes $|\vec{v}|$ and $|\vec{u}|$ respectively. The former is of course just the launch speed s. The latter can be determined from energy conservation:

$$\frac{1}{2}m\,|\vec{u}|^2 = \frac{1}{2}m\,|\vec{v}|^2 + mg\delta, \tag{9}$$

where we understand $\delta > 0$ to be the (vertical) distance of the landing site *below* that of the launch (the case of shot puts, but *not* for basketball shots). Thus,

$$|\vec{u}| = s\sqrt{1 + 2g\delta/s^2} = s\sqrt{1 + \delta/h_{\max}}. \tag{10}$$

In all cases, the condition for maximizing A is $\gamma = 90°$, which leads us to the following conclusion: [11]. Of course, it is also contained in the vector product approach[5].

The initial and final velocities in the trajectory which optimizes the shot put range are orthogonal.

A. Various results

From the geometric solution and the conclusion above, the first consequence is an answer to the original question: What is α_{opt}? Consider the implications of Eqn. (2) for the velocity rectangle: $v_x/v_y = |\vec{v}|\,/\,|\vec{u}|$. Using (10), we arrive at

$$\cot^2\alpha_{opt} = 1 + \frac{\delta}{h_{\max}}, \tag{11}$$

which is essentially the same as the expression often found in the literature: $\sin^2\alpha_{opt} = [2 + 2g\delta/s^2]^{-1}$. The advantage of ours is that the right side is not only manifestly dimensionless, the reduction to the known limits of $\delta = -h_{\max}, 0$, and ∞ is trivial. Here, recall that δ is the distance of the landing site *below* the launch, so that $\delta = -h_{\max}$ means the projectile is stopping *above* the launch site by the maximal amount. For that, the launch condition is obviously "straight up!"

In this connection, we believe it is better to use the y coordinate itself, rather than having to remember that positive δ's go with "below." So, launching from the origin $(x, y) = (0, 0)$, if we want to maximize the range while demanding the projectile lands at y (on the downwards journey), then the optimal angle needed is

$$\cot^2 \alpha_{opt} = 1 - \frac{y}{y_{\max}}, \tag{12}$$

where we have used the notation $y_{\max}$ instead of $h_{\max}$, as $y_{\max}$ immediately signifies the *maximum* y achievable (by an objected launched with a given speed).

Another manifestly dimensionless expression is to use the ratio of the kinetic energies (KE) directly:

$$\cot^2 \alpha_{opt} = \frac{KE_{end}}{KE_{launch}}. \tag{13}$$

The right is, of course, also just $1 + PE_{gain}/KE_{launch}$. All of these manifestly dimensionless forms highlight the physical ratios which control the optimal angle.

Finally, there is another way to express the optimal angle for launch, namely, to provide the appropriate "partition" of the total kinetic energy available at launch ($KE_{launch} = ms^2/2$) into a fraction we can associated with translation in x ($mv_x^2/2$, denoted by KE_x) and the remainder, associated with y ($KE_y \equiv mv_x^2/2$). The resultant optimal "partition" is given by

$$\frac{KE_x}{KE_y} = \frac{KE_{end}}{KE_{launch}}. \tag{14}$$

Note that, as might be expected intuitively, this expression directs us to shoot the projectile more/less vertically if we want to reach a target higher/lower than the launch site. Of course, we recover the equal partition (i.e., $\alpha_{opt} = 45°$) we expect, if the two heights are the same.

IV. NOVEL PERSPECTIVES

In addition to recovering well known results (in new guises), we present some new perspectives of this problem. The simplest is self-evident, though not often stated explicitly: The time reversed trajectory of the shot-put problem is that for the basketball problem (provided they have the same height difference $|\delta|$). The rest of this section is devoted to less obvious, we believe, connections.

A. Duality

Duality is not a precise concept. Yet, examples abound in many areas of both mathematics and physics. Typically, it links two aspects of one problem (or an object) in some complementary manner, while its specific role comes from the precise definition of "complementarity." In physics, wave-particle duality is a well known, century-old notion. In statistical physics, Kramers-Wannier duality[12] appeared some four decades later, while duality links the Kosterlitz-Thouless transition[16] with the roughening of surfaces[21] in the 1980s. The connection between spin systems and lattice gauge theories[17] is another excellent example. More recent and esoteric is the duality between certain string theories of quantum gravity and conformal theories of gauge fields[13]. In mathematics, n-dimensional hypercubes are considered to be dual to $(d-n)$-dimensional ones in d-dimensional space. (e.g., lines being duals to squares in 3 dimensions). Each Platonic solid has a dual partner (though the tetrahedron is "self-dual"). In linear algebra, we have vector spaces and their duals. Fourier transforms are duals, as are Legendre transforms[14]. Many more forms of duality in mathematics are known [15]. One major property in common is that some specific operation is defined to obtain, from some "original" $\mathcal{X}$, its dual $\mathcal{X}^*$. Further, carrying out the operation twice will return one to the original: $\mathcal{X}^{**} = \mathcal{X}$.

Here, let us focus on a much simpler set of concepts of duality, namely, points on a line on a plane. Simplest examples include regarding the pair of points $(x, -x)$ or $(x, 1/x)$ as duals of each other. Clearly, the operations here are "taking the negative of ..." and "taking the inverse of ...", respectively. Combining them also defines a duality relation $(x, -1/x)$, one which is relevant to our shot-put basketball problem.

We have seen that for the optimal trajectory of a projectile, the launch and landing velocities are orthogonal (restricting our considerations to cases where the vertical velocities are of *opposite* signs). In our original setup, the coordinates of the two sites are, for the shot-put problem, $(x, y) = (0, 0)$ and $(\rho, -\delta)$. Of course, this optimal trajectory is part of the parabola

$$y = (\tan\alpha_{opt})\, x - \frac{gx^2}{2\,(s\cos\alpha_{opt})^2}. \tag{15}$$

Here, it is most convenient to cast the parabola in terms of a standard, normalized form

$$\eta = \left(1 - \xi^2\right)/2. \tag{16}$$

(See Fig. 4a and Appendix for details for the explicit transformation from (x, y) to (ξ, η).) In the ξ-η plane, it is clear that the launch/landing site is in the second/fourth quadrant for the shot-put problem (assuming the projectile moves in the $+\xi$ direction). For the basketball case, they are in the third and first quadrants, as illustrated in Fig. 4b. Meanwhile, the direction of the velocities at (ξ, η) are given by the slopes, i.e., $d\eta/d\xi = -\xi$. That the velocities at (ξ, η) and (ξ^*, η^*) are orthogonal means that their slopes are *negative inverses* of each other, i.e.,

$$-\xi = \frac{1}{\xi^*} \implies \xi\xi^* = -1. \tag{17}$$

In this sense, we may consider the launch/landing sites of the optimal trajectory to be "dual" to each other. Of course, as time reversal partners, the shot-put and basketball problems are "dual" to each other also.

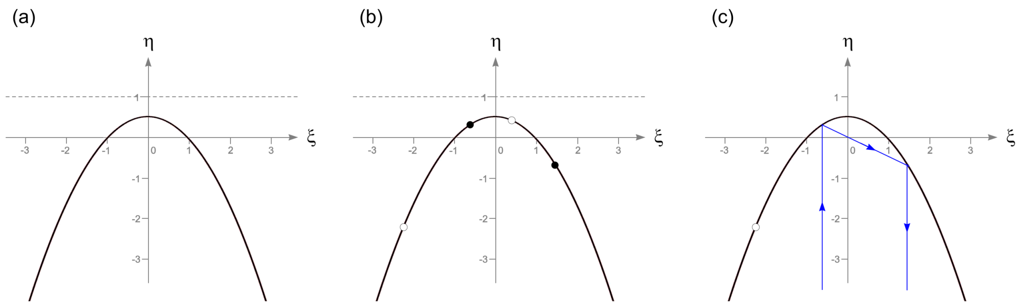

FIG. 4: (a) A standard, normalized parabola (solid line) with it directrix (dashed line). (b) Dual points for the shot put (solid dots) and the basketball (open circles) problem. (c) An incoming beam (dashed line) from below, parallel to the axis, when reflected from a point, travels through the focus to the dual point and becomes an outgoing beam to the axis.

Squaring both sides of Eqn. (17), this duality relationship in η reads

$$(2\eta - 1)(2\eta^* - 1) = 1. \tag{18}$$

Further, the figures give the impression that the line joining the dual points will run through the focus (origin in the ξ-η plane). To prove that, we compute the ratios

$$\frac{\eta}{\xi} = \frac{1}{2}\left[\frac{1}{\xi} - \xi\right]; \quad \frac{\eta^*}{\xi^*} = \frac{1}{2}\left[\frac{1}{\xi^*} - \xi^*\right]. \tag{19}$$

But, due to Eqn. (17), the right sides are identical. This fact implies that the right triangle involving (ξ, η), the origin, and the ξ-axis is similar to the one involving the dual point (ξ^*, η^*), the origin, and the ξ-axis.

Remarkably, there is a connection between our problem and one in simple (geometric) optics. If we place a mirror on the standard, normalized parabola and let a light beam parallel to η-axis incident on any point on it from below, the beam will be reflected to the focus. Thereafter, it will continue to the *dual point* and be reflected as an outgoing beam parallel to the axis. In Fig.3c, we illustrate this optics analogy of the shot-put problem. Of course, the same analysis holds for a beam incident on the dual point so that its trajectory is just the time-reversed version. In this sense, time reversal and duality are usual notions for light beams, just as we may regard the basketball problem as the time reversed or dual of the shot-put problem.

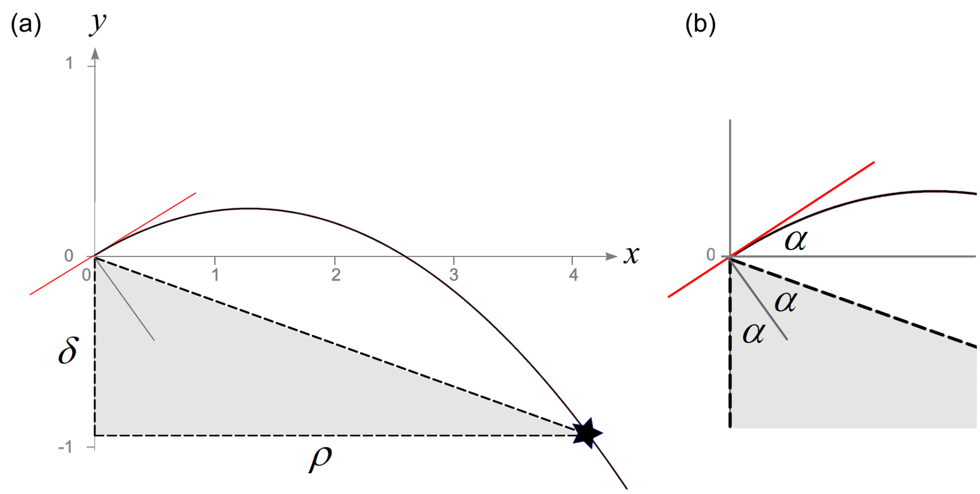

FIG. 5: (a) A projectile is launched at the origin and lands at the star. Note the associated right triangle (shaded) and the trigonometric relation Eqn. (20). The red line is tangent to the trajectory at launch, with the solid black line being its orthogonal. (b) Two angles within the right triangle, both being equal α. (See text.)

Finally, we can draw another conclusion from this optics dual relationship. In Fig. 5a, we show an optimal trajectory and place a right triangle linking the launch/landing points, the horizontal and the y-axis (in the x-y plane). The distances δ and ρ are also shown, so that the ratio ρ/δ is just the tangent of the angle at the top of the triangle. In Fig 5b, recognize the dashed lines are precisely those in the trajectory of the light beam. Meanwhile, the thin solid lines are the tangent and the normal to the parabola at the launch site (also the incident site for our light beam). Therefore, the angles are all α_{opt} (as marked, with subscripts suppressed). So our conclusion from this geometric picture is

$$\tan 2\alpha_{opt} = \frac{\rho_{opt}}{\delta}, \tag{20}$$

a result much more easily obtained than via the tedious algebra implied in Lichentenberg and Wills[3]. Note that, since α_{opt} is a function of (g, s, δ), this result implies we have ρ as a function of (g, s, δ) alone. That will be the topic of the next subsection.

B. Envelope

When the projectile is launched from the origin with the same s and all possible α's, these trajectories will cover a region of the x-y plane. The boundary of this region is referred to as the envelope (of all trajectories). If we can find its functional dependence $y_{env}(x)$, then the "inverse" of it – $x_{env}(y)$ – is clearly the maximum range for the projectile to land at a given height, i.e., $\rho(\delta|g, s)$. Now, the meaning of $y_{env}(x)$ is also clear, as it represents the maximum *height* for the projectile to land for a given range, i.e., $\delta(\rho|g, s)$. But the latter is a much more straightforward problem to solve, in the following sense.

We start with trajectories given by Eqn. (y(x)), but written with the traditional angle α as the variable parameter (and fixed s, g): $y(x) = x \tan\alpha - gx^2/(2s^2 \cos^2\alpha)$. But, for finding α_{opt}, there is no reason to use α instead of

$$\tau \equiv \tan\alpha, \tag{21}$$

as the variable, so that we arrive at a simple quadratic

$$y(x) = x\tau - \frac{x^2}{4h_{\max}}\left(\tau^2 + 1\right). \tag{22}$$

As we vary τ, the maximum clearly occurs at

$$\tau_{opt} = 2h_{\max}/x. \tag{23}$$

In terms of the notations used above, this equation might be written as $\tan\alpha_{opt} = 2h_{\max}/\rho$, which can be regarded as the complement to Eqn. (11).

Substituting (23) back into $y(x)$ gives us the envelope equation:

$$y_{env}(x) = h_{\max} - \frac{x^2}{4h_{\max}}, \tag{24}$$

which can be cast in the form of a standard normalized parabola

$$\frac{y_{env}}{2h_{\max}} = \frac{1}{2}\left[1 - \left(\frac{x}{2h_{\max}}\right)^2\right], \qquad \text{(envelope)}$$

with $(\xi, \eta) = (x, y_{env})/2h_{\max}$! Let us emphasize, this is *not* the normalized parabola of the optimal trajectories discussed in the previous section. The "envelope parabola" here is characterized by just a single parameter ($h_{\max}$), whereas the optimal trajectories are specified by two (any pair from $h_{\max}, \rho, \delta$).

Before closing, let us consider physical arguments which also provides this envelope equation. In this problem, we can expect the fixed quantities g, s not to appear separately. Instead they appear only in the combination $h_{\max} = s^2/2g$, a quantity with units of length. Thus, we can expect $y_{env}(x)$ or its inverse to take the form $\rho(\delta|g, s) = \rho(\delta|h_{\max})$. Further, by dimensional analysis, the relation must appear in the dimensionless (or scaling) form of

$$\frac{y_{env}}{h_{\max}} = f\left(\frac{x}{h_{\max}}\right). \tag{25}$$

Moreover, the envelope must be an even function of x, as the projectile can be launched to the right or the left. Finally, for large x, it is clear that $y_{env} \propto -x^2$. If we now assume that y_{env} is a polynomial function of x, then $f(\chi)$ must be quadratic, i.e., of the form $a + b\chi^2$. Fixing a to unity is trivial, as the maximum height a purely vertical launch can reach is $h_{\max}$. Fixing b comes from considering the $x \to \infty$ limit, where $v_y = 0$, so that $b = 1/4$ and (from $y = -gt^{2/}/2$ and $x = st \implies y = -x^{2/}/(2h_{\max})^2$). The final result is (envelope) again.

To conclude, let us reiterate how this envelope equation can be used in our original problem: Suppose a projectile is launched with speed $\sqrt{2gh_{\max}}$ and various angles. If we want it to get to a target a distance ρ away, then the highest point it can reach is $h_{\max}\left[1 - (\rho/2h_{\max})^2\right]$. If we want it to land a height δ *above* the launch site, then the furthest it can reach is $2h_{\max}\sqrt{1 - \delta/h_{\max}}$. Clearly, the result reduce to the appropriate values in the well known limits.

V. CONCLUDING REMARKS

In this paper, we revisit the ancient problem of finding the optimal angle of release of a projectile (in uniform gravity) for it to land at a specified height above/below the launch site (on its downward journey). We present a simple path to the answer using geometric concepts alone (i.e., without calculus). The main bonus of this route is that, for the optimal trajectory, the velocities at the launch and landing sites are *orthogonal.* Though not entirely new (Refs. 4-8), we believe the details we provided are more "user (student) friendly." Beyond revisiting this issue, we presented two novel perspectives: duality and envelope. In the former, we showed not only the intimate relationship between the launch and landing sites (in the optimal trajectory), we also connect this aspect to certain geometric optics of a parabolic mirror (an incoming beam parallel to the axis of the mirror hitting both dual points before becoming an outgoing beam). In the latter, we demonstrated that the envelope of all trajectories (starting at $x = y = 0$ with the same speed), $y_{env}(x)$, is easily found. Further, it provides precisely the needed information for finding the optimal condition for launch – for either reaching the maximum height (or range), given the target is located at a certain range (or height).

Though neither of the novel aspects are crucial to solving the problem at hand, we believe they serve a different purpose here. As different parts of a physics curriculum often appear disjoint to students (mechanics *vs.* electrodynamics, relativity *vs.* quantum mechanics, optics *vs.* thermodynamics, etc.), it is useful to offer them wider perspectives whenever possible. Thus, we should point out connections across different parts of physics. Examples include renormalization group in quantum field theory and critical phenomena in statistical physics[18], spontaneous symmetry breaking in superconductivity and the Higgs mechanism[19], particle motion in a double-well potential and the density profile of, say, a liquid-vapor interface, and most recently, the quantum Hall effect and El Niño[20]. Indeed, emphasizing links across different areas will encourage students to explore interdisciplinary cross-fertilization. In this context, duality is not only a key concept within one topic (e.g., vector spaces) but also important links between subjects (e.g., lattice gauge theories and spin systems). Similarly, the envelope of trajectories should provide a fresh and simple view of both the general and the original problem of optimization. The crucial lesson for considering these "diversions" is that the wider one's perspective on an issue, the deeper one's insight may develop. Clearly, techniques successful in one area are often effective in another. Thus, building bridges across disciplines should be a valued endeavor.

Let us end by noting that a much more difficult task is to solve the similar problem (optimal trajectories for a projectile) on a spherical earth. Specifically, there are several complications arising from the loss of translational invariance in the "vertical" direction. One is the presence of another dimensionless parameter: the ratio $R_{launch}/R_{landing}$ associated with the radii of the two sites (to the center of gravity). Another is, obviously, the radius of the earth (R_{earth}) though that may be taken as zero for the sake of mathematical simplicity. Finally, there are the possibilities of (a) escape, where the projectile takes off to infinity, and (b) attaining a circular orbit with finite s. As a consequence, we believe there is no simple geometric solution to this generalization of the shot-put problem.

VI. APPENDIX

Any quadratic function $y(x)$

$$y(x) = c_0 + c_1 x + c_2 x^2 \tag{26}$$

can be cast in *standard* form. In physics/engineering terms, let's choose the units of x and y to be the same (e.g., cm), so that the units of c_k are those of x^{1-k}. For our convenience, we will study a downwards pointing parabola (i.e., $c_2 < 0$) here, while $|c_2|$ is half the curvature (y''). Using the freedom to shift the origin of (x, y) and to rescale, we can change variables to (ξ, η) so that the parabola is represented by

$$\eta = \frac{1-\xi^2}{2} \Leftrightarrow \xi^2 = 1 - 2\eta. \tag{27}$$

Note that $\eta - 1/2 = -\xi^2/2$ is not merely in standard form, i.e., $(\eta - \eta_0) = \kappa(\xi - \xi_0)^2/2$, with κ being the curvature. It is also "normalized" in the following sense: The focus of the parabola is located at the origin, $(\xi, \eta) = (0, 0)$, its directrix is at $\eta = 1$, it crosses the ξ-axis at ± 1, and the radius of curvature is 1.

The specifics of the transformation are

$$\xi = \frac{x}{\lambda} - c_1; \quad \eta = \frac{y - c_0}{\lambda} - c_1^2 + \frac{1}{2}, \tag{28}$$

with

$$\lambda = \frac{1}{2(-c_2)}. \tag{29}$$

To check, we see that

$$\frac{1}{2} - \frac{1}{2}\xi^2 = \frac{1}{2} - c_1^2 + c_1 \frac{x}{\lambda} - \frac{1}{2}\left(\frac{x}{\lambda}\right)^2, \tag{30}$$

which is indeed just η, since

$$\frac{y}{\lambda} = \frac{1}{\lambda}\left[c_0 + c_1 x + c_2 x^2\right] = \frac{c_0}{\lambda} + c_1\frac{x}{\lambda} - \frac{1}{2}\left(\frac{x}{\lambda}\right)^2. \tag{31}$$

Acknowledgments

We acknowledge illuminating discussions with K.E. Bassler, E.F. Redish, and T. Takeuchi. We thank Clint Lanham and Pari Sheth for constructive comments, and T. Takeuchi for inviting us to present some of this material at a meeting of the Chesapeake Section of the American Association of Physics Teachers (April, 2022).

---

[1] Galileo Galilei, *Dialogues Concerning TWO NEW SCIENCES*, translated by Henry Crew & Alfonso de Salvio (William Andrew Publishing, Norwich, New York, 2020), p. 275.

[2] William S. Porter, "The range of a projectile,"Phys. Teach. **15**, 258 (1977).

[3] D. B. Lichtenberg and J. G. Wills, "Maximizing the range of the shot put,"Am. J. Phys. **46**, 546–549 (1978).

[4] P. Lucie, "Projectile motion revisited,"Am. J. Phys. **47**, 139–141 (1979).

[5] P. Palffy-Muhoray and D. Balzarini, "Maximizing the range of the shot put without calculus,"Am. J. Phys. **50**, 181 (1982).

[6] S. K. Bose, "Maximizing the range of the shot put without calculus, "Am. J. Phys. **51**, 458-459 (1983).

[7] W. M. Young, "Maximizing the range of the shot put using a simple geometric approach, "Am. J. Phys. **53**, 84 (1985).

[8] Ronald A. Brown, "Maximizing the range of a projectile, "Phys. Teach. **30** 344-347 (1992).

[9] Nicholas P. Linthorne, "Optimum release angle in the shot put,"J. Sports Sci. **19**, 359-372 (2001).

[10] John S. Thomsen, "Maxima and minima without calculus,"Am. J. Phys. **52**, 881-883 (1984).

[11] This conclusion was also reached in References 4 - 8.

[12] H. A. Kramers and G. H. Wannier, "Statistics of the two-dimensional ferromagnet, "Phys. Rev. textbf60 252-–262 (1941).

[13] <https://en.wikipedia.org/wiki/AdS/CFT_correspondence>

[14] R.K.P. Zia, Edward F. Redish, and Susan R. McKay, "Making sense of the Legendre transform,"Am. J. Phys. **77**, 614–622 (2009).

[15] <https://en.wikipedia.org/wiki/Duality_(mathematics)>

[16] J. M. Kosterlitz and D. J. Thouless, "Ordering, metastability and phase transitions in two-dimensional systems,"J. Phys. **C6**, 1181–1203 (1973).

[17] John B. Kogut, "An introduction to lattice gauge theory and spin systems,"Rev. Mod. Phys. **51**, 659–713 (1979).

[18] M. Gell-Mann and F. E. Low, "Quantum Electrodynamics at Small Distances,"Phys. Rev. **95**, 1300–1312 (1954), Kenneth G. Wilson, "Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture,"Phys. Rev. **B4**, 3174–3183 and "Renormalization Group and Critical Phenomena. II. Phase-Space Cell Analysis of Critical Behavior,"Phys. Rev. **B4**, 3184–3205 (1971). For a more accessible account, see e.g., <https://en.wikipedia.org/wiki/Renormalization_group>.

[19] P. W. Anderson, "Plasmons, Gauge Invariance, and Mass,"Phys. Rev., **130**, 439–442 (1963), Peter W. Higgs, "Broken Symmetries and the Masses of Gauge Bosons,"Phys. Rev. Lett. **13**, 508–509 (1964). For more accessible accounts, see e.g., <https://en.wikipedia.org/wiki/Spontaneous_symmetry_breaking> and <https://en.wikipedia.org/wiki/Higgs_mechanism>.

[20] Pierre Delplace, J. B. Marston, Antoine Venaille, "Topological origin of equatorial waves, "Science, **358** 1075–1077 (2017). For a less technical summary, see e.g., <https://phys.org/news/2017-10-earth-climate-topological-insulators-common.html>.

[21] R. K. P. Zia, "Interfacial Problems in Statistical Physics", in *Statistical and Particle Physics: Common Problems and Techniques*, edited by K.C. Bowler and Alan Mckane (Scottish Universities Summer Schools in Physics Publications, Edinburgh, 1984), pp. 247–301.

[22] Strictly they are the same, apart from a trivial rotation by 90° of one of them.