On Gromov-Witten theory of étale gerbes

$\boxed{\text{Set-up}}$ B = complex orbifold (Smooth Deligne-Mumford stack)

G = finite group

$BG = [pt/G]$ = the classifying orbifold of G

G-gerbe over B "=" BG-bundle over B

Construction: $\{U_i\}$ = open cover of B

glueing data: • $\varphi_{ij} \in \mathrm{Aut}(G)$ for each double overlap $U_{ij} := U_i \cap U_j$

• $g_{ijk} \in G$ for each triple overlap $U_{ijk} := U_i \cap U_j \cap U_k$

Compatibility Conditions: • $\varphi_{jk} \circ \varphi_{ij} = \underset{\text{conjugation}}{\mathrm{Ad}}_{g_{ijk}} \circ \varphi_{ik}$ on U_{ijk}

• $g_{jkl}\, g_{ijl} = \varphi_{kl}(g_{ijk})\, g_{ikl}$ on $U_i \cap U_j \cap U_k \cap U_l$

Glueing $\{U_i \times BG\}$ using the above data gives a G-gerbe $\mathcal{Y}$ with a map $\mathcal{Y} \to B$.

$\boxed{\text{Some examples of gerbes}}$

① The trivial G-gerbe: $B \times BG \to B$

② The rth root gerbe: $L \to B$: complex line bundle. $r \in \mathbb{N}$
$(G = \mu_r)$

$\sqrt[r]{L/B}$ = the stack of r^{th} roots of L

(Definition: S = scheme, $\sqrt[r]{L/B}(S) := \left\{ f: S \to B,\ M \in \mathrm{Pic}(S),\ \varphi: f^*L \xrightarrow{\sim} M^{\otimes r} \right\}$)

$\sqrt[r]{L/B} \to B$ is a μ_r-gerbe over B.

$\boxed{\text{The decomposition of a } G\text{-gerbe}}$ (Hellerman–Henriques–Pantev–Sharpe)

$\mathcal{Y} \to B$ = G-gerbe as described above.

The group of Outer automorphisms of G is $\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$

automorphisms given by conjugations

The glueing data $\{\varphi_{ij}, g_{ijk}\}$ satisfy

$$\varphi_{jk} \circ \varphi_{ij} = \underbrace{\mathrm{Ad}_{g_{ijk}}}_{\in \mathrm{Inn}(G)} \circ \varphi_{ik} \quad \text{on } U_{ijk}$$

Define $\phi_{ij} \in \mathrm{Out}(G)$ by
$$\mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$$
$$\varphi_{ij} \longmapsto \phi_{ij}$$

Then $\{\phi_{ij}\}$ satisfy $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ on U_{ijk}

Hence $\{\phi_{ij}\}$ glue $\{U_i \times \mathrm{Out}(G)\}$ to give a principal $\mathrm{Out}(G)$-bundle

$\bar{\mathcal{Y}} \longrightarrow B$, which is called the band of the G-gerbe $\mathcal{Y} \to B$.

$\hat{G}$ = the set of isomorphism classes of finite dimensional irreducible complex representations of G ← finite group.

$\hat{G}$ can be viewed as a finite set of points

$\mathrm{Out}(G)$ acts naturally on $\hat{G}$: $\rho : G \to \mathrm{End}(V_\rho)$ irreducible G-representation
(V_ρ ← finite dim'l vector space)

$\phi \in \mathrm{Aut}(G) \rightsquigarrow \rho \circ \phi : G \to \mathrm{End}(V_\rho)$ irreducible G-representation

If $\phi \in \mathrm{Inn}(G)$, then ρ and $\rho\circ\phi$ are isomorphic as G-representations

So for $[\phi] \in \mathrm{Out}(G)$, $[\rho] \in \hat{G}$, $[\phi]\cdot[\rho] := [\rho\circ\phi]$ defines a right action of $\mathrm{Out}(G)$ on $\hat{G}$.

Definition: $\hat{\mathcal{Y}} := \left[\left(\underset{\text{the band}}{\overline{\mathcal{Y}}} \times \hat{G}\right)/\mathrm{Out}(G)\right] \longrightarrow B$

Because $\hat{G}$ decomposes into $\mathrm{Out}(G)$ orbits, $\hat{\mathcal{Y}}$ decomposes into a disjoint union of components.

$\hat{\mathcal{Y}}$ carries a $U(1)$-gerbe c: For $[\rho] \in \hat{G}$, pick a representative $\rho : G \to \mathrm{End}(V_\rho)$. The obstruction for the family of vector spaces on $\hat{\mathcal{Y}}$

$$\overline{\mathcal{Y}} \times \hat{G} \ni (x, [\rho]) \longmapsto V_\rho$$

to be locally free is a $U(1)$-gerbe c^{-1}. c is obtained from c^{-1}

by $\begin{array}{c} U(1) \to U(1) \\ a \mapsto a^{-1} \end{array}$.

The $U(1)$-gerbe c is flat, and its class in $H^2(\hat{\mathcal{Y}}, U(1))$ is torsion.

In my work with Xiang Tang, we call $(\hat{\mathcal{Y}}, c)$ the dual of the G-gerbe $\mathcal{Y} \to \mathcal{B}$.

My understanding of the decomposition conjecture in Physics is

"Physical theories of $\mathcal{Y}$" $\approx$ "Physical theories of $\hat{\mathcal{Y}}$ twisted by c"

With Xiang Tang, we interpret this mathematically as

"geometry of $\mathcal{Y}$" $\approx$ "geometry of $(\hat{\mathcal{Y}}, c)$"

For this talk: the Gromov-Witten theory aspect of this, namely

Conjecture: Gromov-Witten theory of $\mathcal{Y}$ is equivalent to Gromov-Witten theory of $(\hat{\mathcal{Y}}, c)$

(Kontsevich, Chen-Ruan, Abramovich-Graber-Vistoli)

(Pan-Ruan-Yin)

Related work: Paul Johnson: Conjecture is true for 1-dim'l toric cases.

A partial Status report

To study this Conjecture, there are two things to address:

Ⓐ Identification of state spaces: for Gromov-Witten $(\mathcal{Y})$, the state space is $H^*(I\mathcal{Y})$ inertia stack of $\mathcal{Y}$

For Gromov–Witten $(\hat{\mathcal{Y}}, c)$, the state space is

$$H^*(I\hat{\mathcal{Y}}, \mathcal{L}_c)$$

a system of line bundles on $I\hat{\mathcal{Y}}$ obtained from c.

We need an additive isomorphism

$$H^*(I\mathcal{Y}) \xrightarrow{\simeq} H^*(I\hat{\mathcal{Y}}, \mathcal{L}_c)$$

B Comparison of virtual fundamental classes: We need to somehow compare virtual fundamental classes of moduli spaces of stable maps occuring here, say $\overline{M}_{g,n}(\mathcal{Y}, d)$, $\overline{M}_{g,n}(\hat{\mathcal{Y}}, d)$.

Some known examples:

① For $\mathcal{Y} = B \times BG \to B$ the trivial G-gerbe, we have

$$\hat{\mathcal{Y}} = B \times \hat{G}, \quad c = \text{trivial}.$$

In this case, Conjecture is proved by Andreini–Jiang–T..

A is achieved by direct computations

B is achieved by studying the relationship between $\overline{M}_{g,n}(B \times BG, d)$ and $\overline{M}_{g,n}(B, d)$

② For $\mathcal{Y} = \sqrt[r]{L/B} \to B$ the rth root gerbe, we have

$$\hat{\mathcal{Y}} = B \times \hat{\mu}_r, \quad c = \text{(essentially) trivial}$$

In this case, Conjecture is proved by Andreini-Jiang-T.

A is achieved by direct computations

B is achieved by studying the structure of the map

$$\overline{M}_{g,n}(\mathcal{Y}, d) \longrightarrow \overline{M}_{g,n}(B, d)$$
$$[C \to \mathcal{Y}] \longmapsto [C \to \mathcal{Y} \to B]$$

More general examples:

A can actually be achieved in full generality (Tang-T.):

- From noncommutative geometry, $\mathcal{Y}$ can be viewed as the Morita equivalence class of groupoid algebras associated to presentations of $\mathcal{Y}$

  $(\hat{\mathcal{Y}}, c)$ can be viewed as the Morita equivalence class of c-twisted groupoid algebras associated to presentations of $(\hat{\mathcal{Y}}, c)$

- For a specific groupoid presentation, we can construct a Morita equivalence between the two groupoid algebras.

- A symplectic form on B induces symplectic forms on $\mathcal{Y}$ and $\hat{\mathcal{Y}}$. These can be used to construct deformation quantizations of these groupoid algebras (Tang). The resulting algebras continue to be Morita equivalent.

- The deformation quantized algebras are Morita equivalent, hence they have isomorphic Hochschild cohomology. These cohomology groups are computed to be $H^*(I\mathcal{Y})$ and $H^*(I\mathcal{Y}, c)$ (Neumaier-Pflaum-Posthuma-Tang, Pflaum-Posthuma-Tang-T.)
- The above Morita equivalence is explicit. Hence the above isomorphism & cohomology groups can be made explicit as well (Tang-T.).

(B) can be achieved for gerbes with trivial bands.

For $\mathcal{Y} \to \mathcal{B}$ with the band $\bar{\mathcal{Y}} \to \mathcal{B}$ a trivial $Out(G)$-bundle, we have $\hat{\mathcal{Y}} = \mathcal{B} \times \hat{G}$. On $\mathcal{B} \times [\rho] \subset \hat{\mathcal{Y}}$, the class of c in $H^2(\mathcal{B}, U(1))$ is obtained from the class of $[\mathcal{Y} \to \mathcal{B}]$ in $H^2(\mathcal{B}, Z(G))$ via the map $Z(G) \xrightarrow{\rho} U(1)$.

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To study Gromov-Witten theory of $\hat{\mathcal{Y}}$, we need to study the map $\overline{M}_{g,n}(\mathcal{Y}, d) \to \overline{M}_{g,n}(\mathcal{B}, d)$. This is done by Andreini-Jiang-T. ($\mathcal{B}$ = scheme) and by Tang-T. ($\mathcal{B}$ = stack).