Elliptic genera of pure gauge theories in two dimensions

Richard Eager

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Eric Sharpe

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Decomposition

A two-dimensional theory with a finite global 1-form symmetry decomposes into a disjoint union of theories which individually do not have a 1-form symmetry.

Our motivating examples are two-dimensional gauge theories in which the (finite) center acts trivially.

One-form Symmetries

Two-dimensional pure gauge theories with gauge group G with center Γ have a one-form symmetry $B\Gamma$.

$B\Gamma$ acts by exchanging non-perturbative sectors.

Decomposition for pure gauge theories

A pure gauge theory decomposes into a disjoint union of G/Γ gauge theories with various discrete theta angles

$$\mathcal{T}(G) = \bigoplus_{\theta \in \hat{\Gamma}} \mathcal{T}\left(G/\Gamma\right)_{\theta} .$$

The elliptic genus of a pure G gauge theory is the sum of elliptic genera of pure G/Γ gauge theories with various discrete theta angles.

The elliptic genera of G/Γ gauge theories vanish except for a single discrete theta angle, for which the elliptic genus matches that of the G gauge theory.

We will test decomposition by directly computing the elliptic genus of pure G/Γ gauge theories.

Principal G/Γ bundles

A principal G/Γ bundle on worldsheet T^2 admits a characteristic class we shall denote $w \in H^2(T^2, \Gamma) \cong \Gamma$.

For example, for $SO(k)$ bundles, w is the Stiefel-Whitney class w_2.

Such theories admit analogues of theta angles, known as discrete theta angles, in which the path integral is weighted by phases of the form $\exp(i\theta \cdot w)$ for θ a (log of a) character of Γ, the set of which we shall denote $\hat{\Gamma}$.

Exchanging non-perturbative sectors

In the pure G gauge theory with center Γ, $B\Gamma$ exchanges non-perturbative sectors, taking G/Γ bundles with characteristic classes $w \in H^2(T^2, \Gamma) \cong \Gamma$ to bundles with characteristic class $u \cdot w$ as follows. Cut out a small disk D from T^2 and glue it back in with a transition function $u \in \pi_1(G/\Gamma)$.

Decomposition is a stronger statement than just superselection. For example, only in infinite volume does one get a selection rule from superselection sectors, whereas decomposition holds at finite volume. This distinction is discussed in greater detail in arXiv:1912.01033 [Y. Tanizaki, M. Unsal].

The fields

Pure $\mathcal{N} = (2,2)$ supersymmetric G gauge theory has a vector multiplet consisting of:

- A gauge field A_μ,
- Gauginos λ and $\bar{\lambda}$,
- Scalars $\sigma, \bar{\sigma}$,
- A real auxiliary scalar D.

The gauge field strength is a twisted chiral superfield Σ with lowest component σ.

The action

The Euclidean Yang-Mills Lagrangian $\mathcal{L}_{\mathrm{YM}}$ is

$$\mathrm{Tr}\left(F_{12}^2+D^2+D_\mu\bar{\sigma}D^\mu\sigma+iD[\sigma,\bar{\sigma}]-i\bar{\lambda}\gamma^\mu D_\mu\lambda-i\bar{\lambda}P_+[\sigma,\lambda]-i\bar{\lambda}P_-[\bar{\sigma},\lambda]\right),$$

where

$$P_\pm = \frac{1\pm\gamma_3}{2}.$$

The potential

The classical potential is proportional to

$$\mathrm{Tr}\left[\sigma,\sigma^\dagger\right]^2.$$

The classical vacua occur at the minimum of the potential and satisfy

$$\left[\sigma,\sigma^\dagger\right]=0.$$

Equivalently, the classical Coulomb branch of vacua can be described by the vacuum expectation values of the gauge invariant polynomials in σ.

It is a classical result that this ring of functions is freely generated by rank(G) generators. However, the potential receives quantum corrections, so the IR behavior is potentially more complex.

Aharony et. al. proposed that for G semisimple and simply-connected, the IR theory should be a free theory of twisted chiral multiplets, $Y_i(\Sigma)$, $i = 1, \ldots, \mathrm{rank}(G)$, built out of the generators of the invariant functions on Σ, with axial R-charges r_i given by twice the Casimir degrees[1] d_i of G computed from and in one-to-one correspondence with the possible Casimirs (of which there are as many as the rank).

[1] This follows from the Harish-Chandra isomorphism that relates Casimirs to symmetric invariants.

The contribution of a single twisted chiral multiplet $Y(\Sigma)$ with axial R-charge r to the elliptic genus is

$$\mathrm{Tr}_{RR}\,(-1)^F q^{H_L}\overline{q}^{H_R} y^J \;=\; \frac{\theta_1(\tau|(1-r/2)z)}{\theta_1(\tau|-(r/2)z)},$$

where $q=\exp(2\pi i\tau)$, $y=\exp(2\pi iz)$, J is the left-moving $U(1)_R$ charge, and the genus is computed for periodic left-moving fermions.

Since the low energy theory is a theory of free twisted chiral multiplets, the elliptic genus is expected to be

$$\prod_i \frac{\theta_1(\tau|(1-r_i/2)z)}{\theta_1(\tau|-(r_i/2)z)}.$$

For simply-connected G, this will be demonstrated by explicit computation in [RE].

Gauge group	Dimension	Casimir degrees
$SU(n+1)(A_n)$	$(n+1)^2-1$	$2,3,4,\cdots,n+1$
$\mathrm{Spin}(2n+1)(B_n)$	$n(2n+1)$	$2,4,6,\cdots,2n$
$Sp(2n)(C_n)$	$n(2n+1)$	$2,4,6,\cdots,2n$
$\mathrm{Spin}(2n)(D_n)$	$n(2n-1)$	$n;2,4,6,\cdots,2n-2$
G_2	14	2, 6
F_4	52	2, 6, 8, 12
E_6	78	2, 5, 6, 8, 9, 12
E_7	133	2, 6, 8, 10, 12, 14, 18
E_8	248	2, 8, 12, 14, 18, 20, 24, 30

Casimir degrees for various gauge groups

For example, the elliptic genus of a pure G_2 gauge theory is predicted to be

$$\frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)}\frac{\theta_1(\tau|-5z)}{\theta_1(\tau|-6z)}.$$

Effective central charge

Identifying R-charges $r_i = 2d_i$, we can apply the central charge formula to see that

$$\frac{c_{\rm eff}}{3} \; = \; \sum_i (1 - r_i) \; = \; -\dim G,$$

where $c_{\rm eff}$ is an effective central charge.

It differs from the ordinary central charge as

$$c_{\rm eff} \; = \; c - 24 h_{\rm min},$$

for $h_{\rm min}$ the smallest conformal dimension appearing in the theory, as relevant to theories with continuous spectra.

We can get the same result from the modular transformation properties.

BG

A pure G-gauge theory can be thought of as a sigma model on the stack $BG = [\mathrm{point}/G]$, and this stack has dimension

$$\dim\,[\mathrm{point}/G] \;=\; -\dim G,$$

matching $c_{\mathrm{eff}}/3$ above.

The Lagrangian can be written in (2,2) superspace in the form

$$-\frac{1}{4g^2}\int d^4\theta \mathrm{Tr}\,\overline{\Sigma}\Sigma + \left(-r + i\frac{\theta}{2\pi}\right)\int d\theta^+ d\overline{\theta}^- \;\mathrm{Tr}\Sigma|_{\theta^- = \overline{\theta}^+ = 0} + c.c., \tag{2.1}$$

where Σ is a twisted chiral superfield encoding the gauge field strength, r is a Fayet-Iliopoulos parameter, and θ the theta angle.

In analyzing the low-energy behavior of such theories one often works on the Coulomb branch, along which there is a twisted one-loop effective superpotential which for a pure G/Γ gauge theory with G simply-connected and Γ a subgroup of the center, takes the form

$$W_{\text{eff}} = -\sum_a \Sigma_a \left[-r_a + i\frac{\theta_a}{2\pi} + \frac{1}{|\Gamma|} \sum_{\tilde{\mu}} \alpha^a_{\tilde{\mu}} \left(\ln \left(\sum_b \Sigma_b \alpha^b_{\tilde{\mu}} \right) - 1 \right) \right], \tag{2.2}$$

where now r_a and θ_a are the FI parameters and theta angles for each of the unbroken $U(1)$'s on the Coulomb branch. (No further corrections exist beyond one-loop order.)

The first two terms are the $(-r + i\theta/2\pi)\mathrm{Tr}\Sigma$ of the classical action along the Coulomb branch, and the last is a loop correction, of the same form commonly seen in theories with matter, here ultimately due to W bosons. The $\alpha^a_{\tilde{\mu}}$ are the root vectors of the nonzero roots (indexed by $\tilde{\mu}$) of the Lie algebra of the gauge group. The second term can be simplified, and written as

$$\frac{1}{|\Gamma|}\sum_{\tilde{\mu}} \alpha^a_{\tilde{\mu}} \left(\ln \left(\sum_b \Sigma_b \alpha^b_{\tilde{\mu}} \right) - 1 \right) = \sum_{\tilde{\mu}\ \mathrm{pos}'} \frac{i\pi}{|\Gamma|} \alpha^a_{\tilde{\mu}}, \tag{2.3}$$

giving what amounts to a gauge-group-dependent shift of the theta angle. This was first observed by Hori–Romo. These additional phases will play an important role in our computations of elliptic genera of pure G/Γ gauge theories.

The elliptic genus of a pure G/Γ-gauge theory reduces to a residue integral over the moduli space $\mathcal{M}$ of flat G/Γ-connections on T^2 [F. Benini, R. E., K. Hori, Y. Tachikawa]. Principal G/Γ bundles have a degree-two characteristic class, valued in Γ, which we shall denote $w \in H^2(T^2, \Gamma) \cong \Gamma$, so the moduli space of flat G/Γ connections is a disjoint union of moduli spaces

$$\mathcal{M} = \bigsqcup_{w \in H^2(T^2,\Gamma)} \mathcal{M}_{G/\Gamma, w}.$$

In the sector of bundles with $w = 0$, any G/Γ bundle lifts to a G bundle. Essentially as a result, the elliptic genus of a pure G gauge theory matches that of a pure G/Γ gauge theory in the sector $w = 0$, up to a volume factor $1/|\Gamma \times \Gamma|$ and a Jacobian factor $|\Gamma|^2$:

$$Z(G/\Gamma, w = 0) = \frac{|\Gamma|}{|\Gamma \times \Gamma|} Z(G) = \frac{1}{|\Gamma|} Z(G).$$

[2] This arises from the different normalization of the root systems.

Consider G/Γ gauge theory in a sector in which $w \neq 0$. To describe such bundles, we pick two holonomies p, q around cycles of the torus, which commute up to an element $w \in \Gamma$:

$$pq = wqp.$$

The matrices p and q are the holonomies of any bundle about two cycles of the torus, lifted from G/Γ to G.

These almost-commuting holonomies are the result of lifting commuties holonomies in G/Γ to pairs in G.

Next, we simultaneously diagonalize the adjoint action of p and q on the generators of the Lie algebra in the adjoint representation, writing

$$
\begin{aligned}
pT^\alpha p^{-1} &= \omega_p^\alpha T^\alpha, \\
qT^\alpha q^{-1} &= \omega_q^\alpha T^\alpha,
\end{aligned}
$$

where $\omega_{p,q}^\alpha$ are phases, which enter into the elliptic genus computation.

These phases also appeared in the calculation of the four-dimensional Witten index.

Note that such a diagonalization is not possible for every possible representation in which the T^α may appear; in particular, for the diagonalization above to be possible, one needs for the representation to be acted upon nontrivially by the center detected by p and q. Additionally the phases for the adjoint representation are sufficient to determine the phases for all representations when the center of G/Γ is trivial since the adjoint is a tensor generator of the representation category [Deligne–Milne].

Representations for which such a diagonalization is possible:

Let λ be a dominant weight and write $\lambda = \sum_{i=1}^{l} a_i \varpi_i$ for $a_i \in \mathbb{Z}_{\geq 0}$. Then the irreducible representation of G_{sc} with highest weight λ is faithful precisely in the following cases:

- Type A_l ($l \geq 1$): $\gcd(l+1, a_1 + 2a_2 + \cdots + la_l) = 1$.
- Type B_l ($l \geq 2$): a_l is odd.
- Type C_l ($l \geq 2$): $a_1 + a_3 + a_5 + \cdots$ is odd.
- Type D_l ($l \geq 4$): l is odd and $a_{l-1} + a_l$ is odd
- Type G_2: always
- Type F_4: always
- Type E_6: $a_1 - a_3 + a_5 - a_6$ is not divisible by 3.
- Type E_7: $a_2 + a_5 + a_7$ is odd
- Type E_8: always

(https://mathoverflow.net/questions/328138/non-faithful-irreducible-representations-of-simple-lie-groups)

Representations for which such a diagonalization *not* possible:

Consider $G = SU(2)$, $\Gamma = \mathbb{Z}_2$, with p and q in the 3 of $SU(2)$. It is easy to check that the resulting 3×3 matrices expressing the Lie algebra simply cannot be diagonalized with respect to nontrivial p and q.

If the phases $\omega_{p,q}$ are different from one, then, those 'directions' in the group are fixed. If they are equal to one, on the other hand, then the group is unconstrained in those directions, and so one must integrate over corresponding Wilson lines, over the corresponding moduli space of flat connections, to get the elliptic genus.

Schweigert and Borel–Friedman–Morgan describe the moduli spaces:

$$\mathcal{M}_{G/\Gamma,w} = \mathcal{M}_{\tilde{G}(w),1}$$

for some other group $\tilde{G}(w)$ that depends upon G/Γ and w, where $\mathcal{M}$ denotes the moduli space of flat connections.
Roughly speaking, we can think of the groups $\tilde{G}(w)$ as being obtained by folding the affine Dynkin diagram according to the action of $w \in \Gamma$.

G/Γ	w	$\tilde{G}(w)$
$A_{n-1} \sim SU(n)/\mathbb{Z}_n$	d	$SU(m),\ m = gcd(n,d)$
$B_n \sim \mathrm{Spin}(2n+1)/\mathbb{Z}_2$	1	$Sp(2n-2)$, $\mathrm{Spin}(2n-1)$
$C_{2n} \sim Sp(4n)/\mathbb{Z}_2$	1	$Sp(2n)$, $\mathrm{Spin}(2n+1)$
$C_{2n+1} \sim Sp(4n+2)/\mathbb{Z}_2$	1	$Sp(2n)$, $\mathrm{Spin}(2n+1)$
$D_{2n+1} \sim \mathrm{Spin}(4n+2)/\mathbb{Z}_4$	1	$Sp(2n-2)$, $\mathrm{Spin}(2n-1)$
	2	$Sp(4n-2)$, $\mathrm{Spin}(4n-1)$
	3	$Sp(2n-2)$, $\mathrm{Spin}(2n-1)$
$D_{2n} \sim \mathrm{Spin}(4n)/\mathbb{Z}_2 \times \mathbb{Z}_2$	$(1,0)$	$Sp(2n)$, $\mathrm{Spin}(2n+1)$
	$(0,1)$	$Sp(4n-4)$, $\mathrm{Spin}(4n-3)$
	$(1,1)$	$Sp(2n)$, $\mathrm{Spin}(2n+1)$
$E_6/\mathbb{Z}_3$	1	G_2
	2	G_2
$E_7/\mathbb{Z}_2$	1	F_4

List of groups $\tilde{G}(w)$ whose moduli space of flat connections matches that of a moduli space of flat G/Γ connections with nontrivial characteristic class $w \in H^2(T^2, \Gamma)$.

To describe the moduli spaces $\mathcal{M}_{G/\Gamma,w=0}$ more concretely, let T a maximal torus of G/Γ [3] with corresponding Cartan subalgebra $\mathfrak{h}$. Let Q be the root lattice, P be the weight lattice, and Λ_{char} be the character lattice of G/Γ. Similarly, let $Q^\vee$ be the coroot lattice, $P^\vee$ be the coweight lattice, and $\Lambda^\vee_{char}$ be the co-character lattice. Then the Cartan torus of G/Γ can be identified with $\mathfrak{h}/2\pi\Lambda^\vee_{char}$. The center of and fundamental groups of G/Γ are

$$Z(G/\Gamma) \cong P^\vee/\Lambda^\vee_{char} \cong \Lambda_{char}/Q,$$
$$\pi_1(G/\Gamma) \cong \Lambda^\vee_{char}/Q^\vee \cong P/\Lambda_{char}.$$

[3] Not to be confused with the elliptic curve T^2.

Let

$$\mathfrak{M} = \mathfrak{h}_{\mathbb{C}}/(\Lambda^{\vee}_{char} + \tau\Lambda^{\vee}_{char}) \ ,$$

then the moduli space of flat G/Γ-connections on T^2 with $w = 0$ is

$$\mathcal{M}_{G/\Gamma, w=0} = \mathfrak{M}/W,$$

where W is the Weyl group of G/Γ.

For G simply-connected the cocharacter lattice is equal to the coroot lattice. In the opposite extreme of G/Γ with trivial center, the cocharacter lattice is equal to the coweight lattice. The relations between the cocharacter lattices mean that the moduli space $\mathcal{M}_{G,1}$ is an order $|\Gamma \times \Gamma|$ cover of $\mathcal{M}_{G/\Gamma,w=0}$.

The elliptic genus of a pure G/Γ theory (with bundles of vanishing characteristic class) is given by

$$Z_{T^2}(\tau, z, w=0) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \underset{u=u_*}{\text{JK-Res}}\big(\mathrm{Q}(u_*), \eta\big) \; Z_{\text{1-loop}}(\tau, z, u)$$

where $|W|$ is the order of the Weyl-group of G.
The parameter $q = e^{2\pi i \tau}$ in $Z_{\text{1-loop}}$ specifies the complex structure of the torus T^2 and $y = e^{2\pi i z}$ is the fugacity for the left-moving $U(1)$ R-symmetry. The coordinates u_a on the moduli space $\mathfrak{M}$ can equivalently be described by the coordinates $x_a = e^{2\pi i u_a}$.

The contribution of a vector multiplet V with gauge group G/Γ to $Z_{\text{1-loop}}$ for the $w = 0$ characteristic class is

$$Z_{V,G/\Gamma}(\tau, z, u) = \left(\frac{2\pi\eta(q)^3}{\theta_1(q, y^{-1})}\right)^{\text{rank } G} \prod_{\alpha \in G} \frac{\theta_1(q, x^\alpha)}{\theta_1(q, y^{-1}x^\alpha)} \prod_{a=1}^{\text{rank } G} \mathrm{d}u_a \ .$$

The product is over the roots α of the gauge group and $\eta(q)$ is the Dedekind eta function.

For bundles with non-trivial characteristic classes w, the contribution to $Z_{\text{1-loop}}$ is modified. Using the eigenvalues $\omega^{\alpha}_{p,q}$, one can then construct an elliptic genus for bundles of fixed characteristic class w as a product of ratios

$$\frac{\theta_1(\tau|v_\alpha)}{\theta_1(\tau|-z+v_\alpha)},$$

for nonzero v_α, where

$$v_\alpha = \ln\frac{\omega^\alpha_p}{2\pi i} + \tau\ln\frac{\omega^\alpha_q}{2\pi i},$$

The residue integral takes the form

$$\left(\frac{2\pi\eta(q)^3}{\theta_1(q,y^{-1})}\right)^{\text{rank }\tilde{G}(w)} \prod_{\alpha\in G} \frac{\theta_1(\tau|v_\alpha)}{\theta_1(\tau|-z+v_\alpha)} \prod_{a=1}^{\text{rank }\tilde{G}(w)} \mathrm{d}u_a \ .$$

for every vanishing v. The resulting residue integral is computed as a Jeffrey-Kirwan residue over (a cover of) the moduli space of those flat connections preserving the holonomy.

Finally, we will combine them, to form the elliptic genus as a function of the discrete theta angle. These different contributions are each weighted with potentially two different phases. First, there is a factor $\exp(i\theta \cdot w)$, where $\theta \in \hat{\Gamma}$ is a choice of discrete theta angle. Second, there is a factor of the form $\exp(iw \cdot t)$, where

$$t_a = -\frac{\pi i}{|\Gamma|} \sum_{\tilde{\mu} \text{ pos}'} \alpha^a_{\tilde{\mu}}, \tag{3.1}$$

and w is encoded in w_a so that

$$t \cdot w = \sum_a t_a w_a. \tag{3.2}$$

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[4]Strictly speaking, the t_a are not uniquely defined, as there are e.g. branch cut ambiguities, but the phase factor above is well-defined, as discussed in detail in [**?**]. Put another way, the t_a encode a constant shift, due to quantum corrections, to the discrete theta angle θ.

Thus, if we label the contribution to the elliptic genus of a pure G/Γ gauge theory in a sector with bundles of characteristic class w by $Z(G/\Gamma, w)$, then the elliptic genus for a general characteristic class has the form

$$Z(G/\Gamma, \theta) = \sum_{w} \exp(iw \cdot \theta) \exp(iw \cdot t) Z(G/\Gamma, w). \tag{3.3}$$

The elliptic genus of pure $SU(2)$ gauge theory is

$$\frac{1}{2}\sum_{u_* \in \mathfrak{M}^+_{\text{sing}}} \frac{i\eta(q)^3}{\theta_1(\tau|-z)} \oint_{u_*} \mathrm{d}u \, \frac{\theta_1(\tau|2u)}{\theta_1(\tau|-z+2u)} \frac{\theta_1(\tau|-2u)}{\theta_1(\tau|-z-2u)},$$

where the contributing poles are located at

$$\mathfrak{M}^+_{\text{sing}} = \left\{ \frac{z}{2}, \frac{z+1}{2}, \frac{z+\tau}{2}, \frac{z+\tau+1}{2} \right\}.$$

Elliptic genera of pure $SO(3)$ gauge theories were computed in [Kim–Kim–Park] The authors argued that the pure $SU(2)$ and the $SO(3)_-$ theories have the same elliptic genus, given by

$$\frac{\theta_1(\tau|-z)}{\theta_1(\tau|-2z)} = \frac{1}{2}\frac{\theta_1(\tau|+1/2)}{\theta_1(\tau|-z+1/2)}\frac{\theta_1(\tau|+\tau/2)}{\theta_1(\tau|-z+\tau/2)}\frac{\theta_1(\tau|-(1+\tau)/2)}{\theta_1(\tau|-z-(1+\tau)/2}$$

while the elliptic genus of the pure $SO(3)_+$ theory vanishes identically.

This is consistent with the results of Gu et. al., which argued that in pure $SO(3)$ gauge theories, only for the nontrivial discrete theta angle are there supersymmetric vacua, and supersymmetry is broken in the IR in $SO(3)_+$. It is also consistent with decomposition [Hellerman et. al., Sharpe], which in this case can be schematically expressed as

$$SU(2) = SO(3)_+ + SO(3)_-. \tag{4.1}$$

Gauge group	Discrete theta angle for which susy unbroken
$SU(k)/\mathbb{Z}_k$	$-(1/2)k(k-1) \mod k$
$\mathrm{Spin}(2k+1)/\mathbb{Z}_2$	$1 \mod 2$
$\mathrm{Spin}(4k)/\mathbb{Z}_2 \times \mathbb{Z}_2$	$k(2k-1) \mod 2,\ 0 \mod 2$
$\mathrm{Spin}(4k+2)/\mathbb{Z}_4$	$2k(2k-1) \mod 4$
$Sp(2k)/\mathbb{Z}_2$	$(1/2)k(k+1) \mod 2$
$E_6/\mathbb{Z}_3$	$0 \mod 3$
$E_7/\mathbb{Z}_2$	$1 \mod 2$

List of distinguished discrete theta angles for various non-simply-connected gauge groups, for which a pure gauge theory admits supersymmetric vacua, summarizing results from E. Sharpe, W. Gu, E. Sharpe, H. Zou [arXiv:2005.10845]

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Thank you!