Decomposition in orbifolds with discrete torsion

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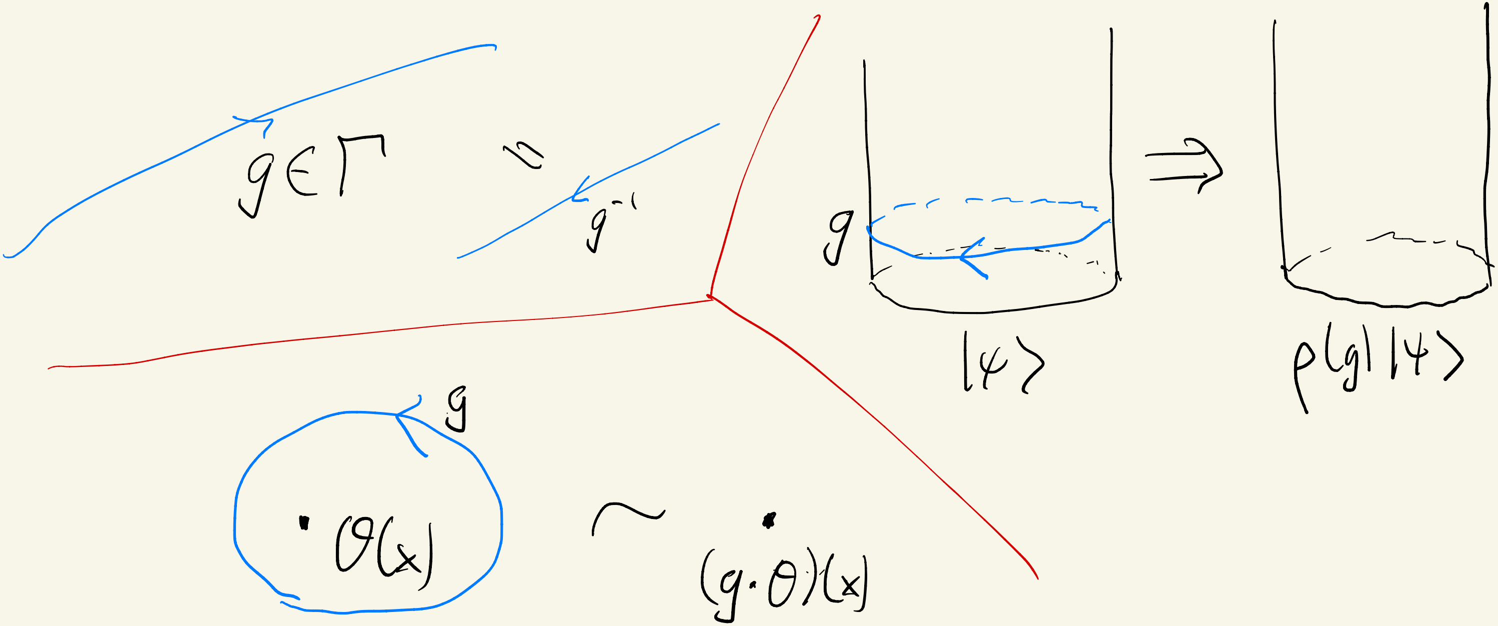
Daniel Robbins, University at Albany

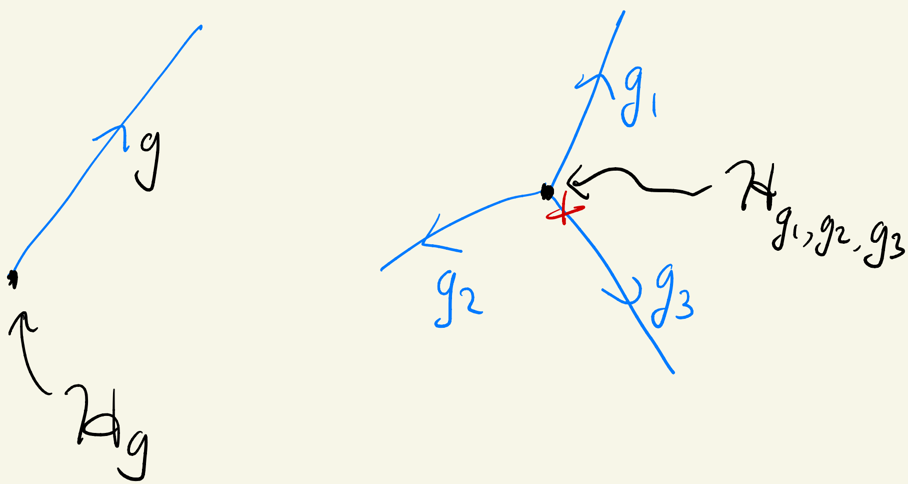
based on: 2101.11619 w/ E. Sharpe
T. Vandermeulen

- Start with a 2D CFT with a (finite) global symmetry group Γ.

- We can think about the action of Γ in terms of Topological Defect Lines (TDLs)



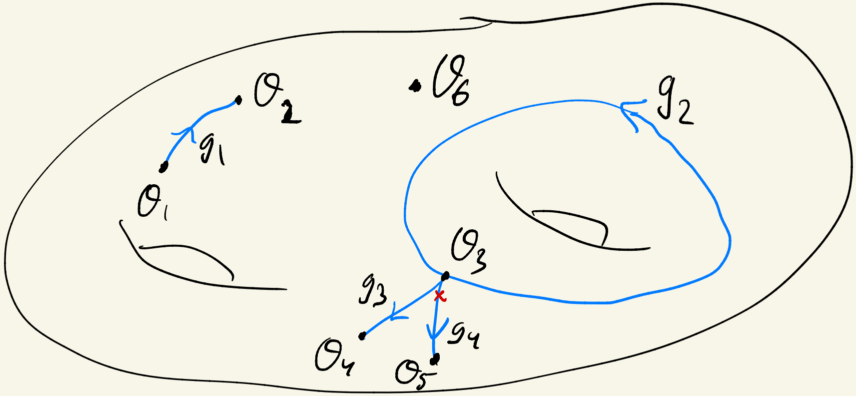
- TDLs can meet at junctions, and for each possible junction there will be a space of operators which can be inserted.



"g-twisted sector Hilbert space"

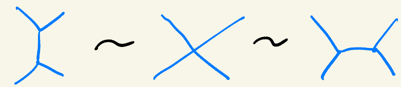
(see 1802.04445, Chang, Lin, Shao, Wang, + Yin)

Correlation functions with TDLs:



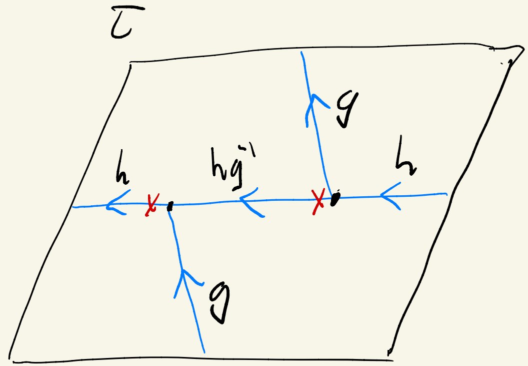
- Lines can be deformed without changing corr. fn., but vertices can't unless inserted operator is weight zero.

- For effective symmetries, $\mathcal{H}_{g_1,\dots,g_n}$ contains weight zero states iff $g_1 g_2 \cdots g_n = 1$.

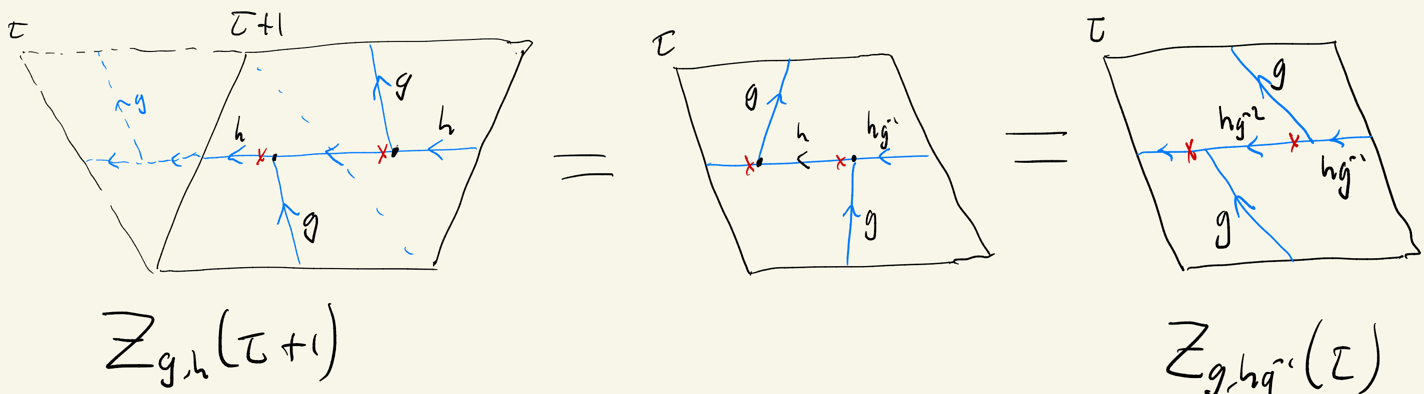


- For anomaly-free Γ, can also freely resolve junctions

· "Partial Traces" $Z_{g,h}(\tau,\bar{\tau})$, $gh=hg$.



· Modular transformations: $Z_{g,h}(\tau+1) = Z_{g,hg^{-1}}(\tau)$, $Z_{g,h}(-1/\tau) = Z_{h,g^{-1}}(\tau)$



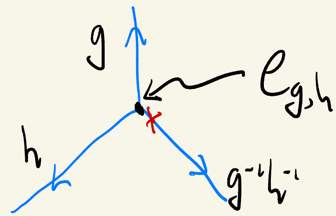
- A network of these TDLs can be thought of as a background Γ gauge field.

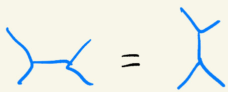
- If we want to gauge Γ (construct orbifold), we should sum over all background configurations.

- For instance, one-loop Γ-orbifold partition function,

$$Z^{\Gamma}(\tau,\bar{\tau}) = \frac{1}{|\Gamma|} \sum_{gh=hg} Z_{g,h}(\tau,\bar{\tau})$$

- Implicitly, we are inserting some weight zero vectors $e_{g,h} \in \mathcal{H}_{g,h,g^{-1}h^{-1}}$. We can make a new choice $e'_{g,h} = \omega_{g,h} e_{g,h}$



- Consistency () implies

$$\omega(g_2,g_3)\,\omega(g_1,g_2g_3) = \omega(g_1g_2,g_3)\,\omega(g_1,g_2)$$

- Shifting $\omega \to \omega'(g,h) = \omega(g,h)\,\mu(g)\,\mu(h)\,\mu(gh)^{-1}$

$$\Longrightarrow \quad \omega \in H^2(\Gamma, U(1))$$

- $Z_{g,h}(\tau) \to Z_{g,h}(\tau) \cdot \dfrac{\omega(g,h)}{\omega(h,g)}$ $\rightsquigarrow$ discrete torsion.

$$\epsilon(g,h) = \frac{\omega(g,h)}{\omega(h,g)}$$

- Now consider the case when a subgroup $K \subseteq \Gamma$ acts non-effectively (on local operators)

- K must be normal $\left(gkg^{-1}|\psi\rangle = g \cdot k(g^{-1}|\psi\rangle) = g \cdot (g^{-1}|\psi\rangle) = |\psi\rangle \right.$
$\left. \Rightarrow gkg^{-1} \text{ acts trivially} \right)$

Define $G = \Gamma/K$.

SES: $1 \longrightarrow K \xrightarrow{\imath} \Gamma \xrightarrow{\pi} G \longrightarrow 1$

- Now $\mathcal{H}_{g_1,\dots,g_n}$ contains weight zero states iff $g_1 g_2 \cdots g_n \in K$.

- The Γ-partial traces $Z^{\Gamma}_{\gamma_1,\gamma_2}$ are expected to be related to G-partial traces. We'll take*

$$Z^{\Gamma}_{\gamma_1,\gamma_2} = Z^{G}_{\pi(\gamma_1),\pi(\gamma_2)}$$

Then $$Z^{\Gamma} = \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2=\gamma_2\gamma_1} \epsilon(\gamma_1,\gamma_2)\, Z^{\Gamma}_{\gamma_1,\gamma_2}$$

$$= \frac{1}{|G|} \sum_{g,h\in G} \left(\frac{1}{|K|} \sum_{\substack{\gamma_1\gamma_2=\gamma_2\gamma_1 \\ \pi(\gamma_1)=g \\ \pi(\gamma_2)=h}} \epsilon(\gamma_1,\gamma_2) \right) Z^{G}_{g,h}$$

* until end of the talk

- Consider first $K = \Gamma$ (orbifold of a point)

  Each $\mathcal{H}_k \cong \mathcal{H}_1 = \mathcal{H}$. For every $\mathcal{O} \in \mathcal{H}$, maps under isomorphism to $\mathcal{O}^{(k)} \in \mathcal{H}_k$.

- If there is no discrete torsion $(\epsilon(\gamma_1, \gamma_2) = 1)$, then

$$Z^{\Gamma} = \mathcal{N} \, Z^{\{1\}}_{1,1}, \quad \text{where } \underline{\mathcal{N} := \frac{1}{|\Gamma|} \sum_{\gamma_1\gamma_2 = \gamma_2\gamma_1} 1} = \underline{\#\text{ of conj. classes of } K} = \underline{|\hat{K}|},$$

where $\hat{K}$ is the set of isomorphism classes of irreducible representations of K.

- For example, if K is abelian, then $\mathcal{N} = \frac{|K|^2}{|K|} = |K|$.

- For an example of K nonabelian, take $K = S_3$.

$$\begin{array}{lcr} \text{Commuting pairs:} & (1, g) & 6 \times 1 \\ & ((12), 1), ((12), (12)) & 2 \times 3 \\ & ((123), 1)\ ((123), (123)), ((123), (132)) & \underline{3 \times 2} \\ & & 18. \end{array}$$

$$\Rightarrow Z^{S_3} = 3 Z^{\{1\}}_{\omega 1} = 3 Z$$

- How does discrete torsion change the story?

- Take $K = \Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, a, b, ab\}$.

$$H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2 .$$

Nontrivial element $\rightsquigarrow$ $\epsilon(a,b) = \epsilon(a,ab) = \epsilon(b,a)$
$= \epsilon(b,ab) = \epsilon(ab,a) = \epsilon(ab,b) = -1,$
all other $\epsilon(g,h) = 1$.

$$Z_\omega^{\mathbb{Z}_2\times\mathbb{Z}_2} = \frac{1}{4}\underbrace{\Big(\sum_{\gamma_1,\gamma_2\in\mathbb{Z}_2^2}\epsilon(\gamma_1,\gamma_2)\Big)}_{10\times(+1)+6\times(-1)=4} Z_{1,1}^{\{1\}} = Z$$

$\rightsquigarrow$ equivalent to the parent theory!

- In general, $\mathcal{N}_\omega := \frac{1}{|K|}\sum_{k_1k_2=k_2k_1}\epsilon(k_1,k_2) = |\hat{K}_\omega|$,

where $\hat{K}_\omega$ is the set of isomorphism classes of irreducible ω-projective representations of K.

- ω-projective representation of K for $\omega \in Z^2(K, U(1))$

  is a vector space V and a map $\rho : K \to GL(V)$

  satisfying $\rho(k_1)\rho(k_2) = \omega(k_1,k_2)\,\rho(k_1k_2)$

- e.g. for $K = \mathbb{Z}_2 \times \mathbb{Z}_2$

  $\omega(a,b) = \omega(b,ab) = \omega(ab,a) = i$, $\quad \omega(b,a) = \omega(ab,b) = \omega(a,ab) = -i$

  Up to isomorphism, only projective irrep is

  $$\rho(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho(ab) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

  i.e. the Pauli matrices.

· What if K is not all of Γ?

Then Z^{Γ} should decompose into orbifolds by subgroups of G.

· Ex 1. $1 \to \underset{\{1,a\}}{\mathbb{Z}_2} \to \underset{\{1,a,b,ab\}}{\mathbb{Z}_2 \times \mathbb{Z}_2} \to \underset{\{\bar{1},\bar{b}\}}{\mathbb{Z}_2} \to 1$

i) No discrete torsion:

$$Z^{\mathbb{Z}_2\times\mathbb{Z}_2} = \frac{1}{4}\left(4 Z^{\mathbb{Z}_2}_{\bar{1},\bar{1}} + 4 Z^{\mathbb{Z}_2}_{\bar{1},\bar{b}} + 4 Z^{\mathbb{Z}_2}_{\bar{b},\bar{1}} + 4 Z^{\mathbb{Z}_2}_{\bar{b},\bar{b}}\right)$$

$$= 4 Z^{\mathbb{Z}_2}$$

$\rightsquigarrow$ 4 disjoint copies of the $\mathbb{Z}_2$ orbifold

ii) With d.t: $Z^{\mathbb{Z}_2\times\mathbb{Z}_2}_{\omega} = \frac{1}{4}\left(4 Z^{\mathbb{Z}_2}_{\bar{1},\bar{1}} + (2-2)\left(Z^{\mathbb{Z}_2}_{\bar{1},\bar{b}} + Z^{\mathbb{Z}_2}_{\bar{b},\bar{1}} + Z^{\mathbb{Z}_2}_{\bar{b},\bar{b}}\right)\right)$

$= Z$ $\rightsquigarrow$ <u>one</u> copy of the parent theory.

· Ex 2. $1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$.

$$H^2(D_4, U(1)) \cong \mathbb{Z}_2$$

i) without d.t: $Z^{D_4} = Z^{\mathbb{Z}_2 \times \mathbb{Z}_2} + Z_\omega^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ discrete torsion in $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

ii) with d.t: $Z_\omega^{D_4} = Z^{\mathbb{Z}_2}$ a $\mathbb{Z}_2$ subgroup of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$

<u>Ex. 3</u> $1 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to S_4 \to S_3 \to 1$

$$H^2(S_4, U(1)) \cong \mathbb{Z}_2$$

i) without d.t: $Z^{S_4} = Z + Z^{S_3}$

ii) with d.t: $Z_\omega^{S_4} = Z^{S_3}$

- Conjectured algorithm:

$$
\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "\imath"] & \Gamma \arrow[r, "\pi"] & G \arrow[r] \arrow[l, bend right, "s"'] & 1
\end{tikzcd}
$$

section, $\pi(s(g)) = g$

  - Given $1 \to K \xrightarrow{\imath} \Gamma \xrightarrow{\pi} G \to 1$

    and $\omega \in H^2(\Gamma, U(1))$:

  - Construct $\hat{K}_{\imath^*\omega}$
  - Construct an action of G on $\hat{K}_{\imath^*\omega}$:

$$(L_g\varphi)(k) = \frac{\omega(s^{-1}k, s)}{\omega(s, s^{-1}k)}\varphi(s^{-1}ks) \qquad s = s(g)$$

Not a group action on φ's, but is on isomorphism classes.

  - Decompose $\hat{K}_{\imath^*\omega}$ into G-orbits with representatives $[\rho_a]$ and $G_a = \mathrm{Stab}([\rho_a]) \subseteq G$.

- Each orbit gives a G_a orbifold in decomposition with discrete torsion $\omega_a \in Z^2(G_a, U(1))$

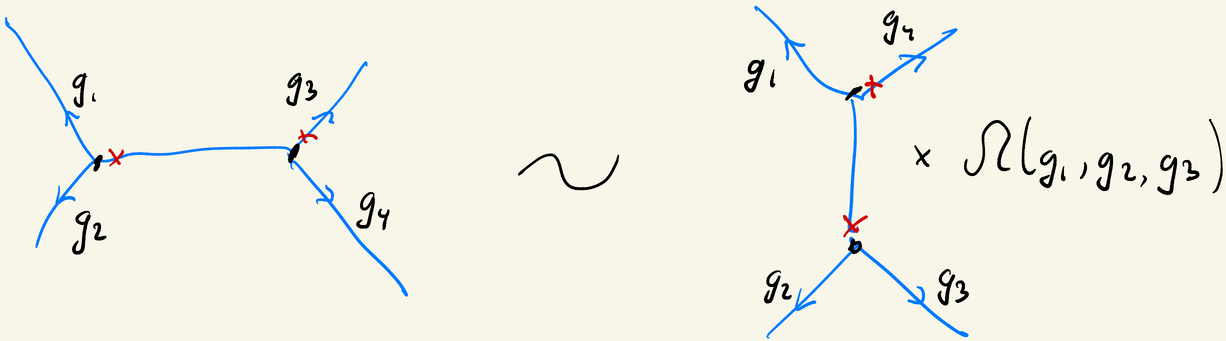
- ω_a is determined by constructing from ρ_a a proj. rep $\tilde{\rho}_a$ of Γ with $\tilde{\omega}_a = \omega / \pi^* \omega_a$

  (see paper for details)

This algorithm works in all examples we checked.

Anomalies. For a CFT with symmetry G, the potential anomaly lives in $H^3(G, U(1))$.

How do we see its effects?



Modular transformations of $Z_{g,h}$ pick up phases.

$G = \mathbb{Z}_2.$ $\quad H^3(\mathbb{Z}_2, U(1)) \cong \mathbb{Z}_2$

$$Z_{1,0}(\tau+2) = -Z_{1,0}(\tau)$$

Resolution: $\quad 1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \mathbb{Z}_4 \xrightarrow{\pi} \mathbb{Z}_2 \longrightarrow 1$

$$[\pi^* \Omega] = 0 \in H^3(\mathbb{Z}_4, U(1))$$

$$Z^{\Gamma}_{\gamma_1,\gamma_2} = \underline{\epsilon(\gamma_1,\gamma_2)}\, Z^{G}_{\pi(\gamma_1),\pi(\gamma_2)}$$

not from discrete torsion here!

$$B(g,k) \in Z^1(G, H^1(K, U(1)))$$

$$\epsilon(\gamma_1,\gamma_2) = \frac{B(\pi(\gamma_2),\ s(\pi(\gamma_1))^{-1}\gamma_1)}{B(\pi(\gamma_1),\ s(\pi(\gamma_2))^{-1}\gamma_2)}$$

Interpretation as a generalization of quantum symmetry.

Thanks!