A GLSM view on Homological Projective Duality (HPD)

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Outline:
- GLSM's and B-brane categories
- HPD (of a projective var. X) from physics view point
- Examples

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GLSM: (2,2) 2d gauge theory $\left(G,\ \rho_m : G \to \underline{GL(V)},\ W : V \to \mathbb{C},\ R\right)$ — gauge invt.; vector R-charge

$\left(\begin{matrix} \cong \\ U(1)^s \end{matrix}\right)$

Two set of parameters: i) $t = \underbrace{\zeta_0}_{\text{bare}} - i\theta + b_1 \ln\left(\frac{\mu}{\Lambda}\right) \in \mathrm{Lie}(T_G^{\vee})^{\mathbb{C}}$

b_1: const. vector $(\Lambda^N V)$; μ: energy scale; Λ: UV cut-off

FI-theta parameters = stringy Kähler moduli $\mathcal{M}_K$

(low energy (IR) $\mu \to 0$)

ii) $\mathrm{Coeffs}(W)/\sim$ = cpx. str. moduli $\mathcal{M}_{CS}$

We have the following spaces associated w/ GLSM $(G, \rho_m, \dots)$

$\mu : V \longrightarrow \mathrm{Lie}(G)^{\vee} \rightsquigarrow$ moment map assoc. to $\rho_m \simeq$ D-terms

$$Y_\zeta := \mu^{-1}(\zeta)/G \qquad \zeta = \zeta(\mu)$$

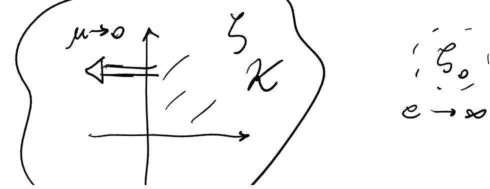
$$X_\zeta := Y_\zeta \cap dW^{-1}(0) \qquad (= \text{"classical" Higgs branch})$$

Rmk: $\rho_m : G \to \underline{\underline{SL(V)}}$ "CY GLSM" $\to$ anomaly free (of Axial R-charge), so $b_1 = 0$

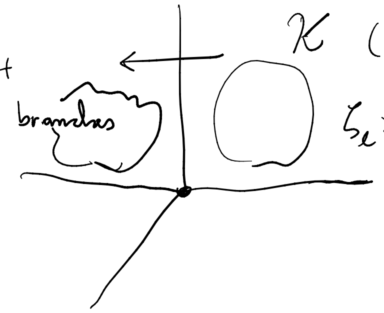
$GL(V) \to \underline{\text{anomalous}}$

Assumption: $\exists$ a region $\mathcal{K}$ in ζ-space where X_ζ is compact and smooth (i.e. in the region $\mathcal{K}$, in the gauge decoupling limit, the theory can be approximated by a NLSM w/ target space X) $\Leftrightarrow \exists$ a geometric phase

We denote this GLSM by $\mathcal{T}_X$.



B-branes category



each phase, we can study the SUSY preserving boundary cdts., $2_B \subset (2,2) \rightsquigarrow$ B-branes form a (triangulated) category

w/ our assumptions $X \xhookrightarrow{f} \mathbb{P}^m = \mathbb{P}(V_X)$ $(\neq V)$ then the functions $f_0, \dots, f_m$ have charge

$[f_0(\phi), \dots, f_m(\phi)]$
belong to $V^\vee$

$\underline{Q} \in \mathbb{Z}_{>0}$ under $U(1)_e \subset G = G/_{U(1)_e} \times U(1)_e$, we write the FI parameter assoc. w/ $U(1)_e$ as ζ_e, then the B-brane categories of both phases $\zeta_e \gg 1$ and $\zeta_e \ll -1$ are related as follows

$$\to D(X_{\zeta_e \gg 1}) = \langle \underbrace{\boxed{D(Y_{\zeta_e \ll -1}; W_{\zeta_e \ll -1})}}_{\substack{\text{Cat. of B-branes on} \\ \text{the Higgs branch of} \\ \zeta_e \ll -1}}, \underbrace{E_1, \dots, E_k}_{\substack{\text{Contribution from} \\ \text{the Coulomb branch} \\ (\simeq \text{ isolated massive} \\ \text{vacua, e.g.})}} \rangle$$

$W|_{Y_{\zeta_e \ll -1}}$

$D(X_{\zeta_e \gg 1}) = D^b\mathrm{Coh}(X)$

Semiorthogonal decomposition (SOD)

e.g. $X = \mathbb{P}^m$ $(G = U(1), \rho_m : U(1) \to GL(\mathbb{C}^{m+1}), W \equiv 0, R = 0)$

$\zeta \ll 1$

$$\to D\,\mathrm{Coh}(\mathbb{P}^m) = \langle \overbrace{\underbrace{E_0, \dots, E_{m+1}}_{\text{pure Coulomb}}}^{\zeta \ll -1} \rangle = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(m) \rangle$$

SOD

Given $\underline{T_X}$ we construct a (extended) GLSM we call $T_{\mathcal{X}}$

$$T_{\mathcal{X}} = (\hat{G} = G \times U(1)_e, \hat{\rho}_m : \hat{G} \to GL(V \oplus V'), \hat{W}, \hat{R})$$

V' is a rep. of $U(1)_0 \times U(1)_e \subset \hat{G}$ of weights $(-1, -\underline{Q}) \oplus (1,0)^{\oplus (\underline{m}+1)}$

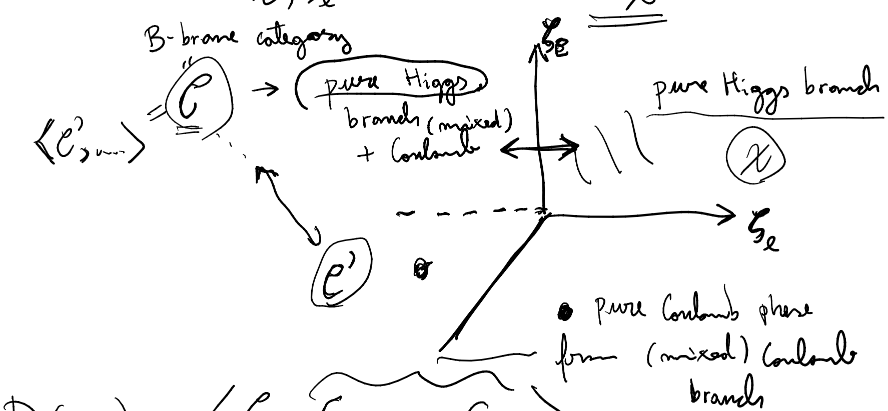
V' is a rep. of $U(1)_\ell \times U(1)_e \subset \hat{G}$ of weights $(-1, -\underset{\sim}{Q}) \oplus (1,0)^{\oplus (\underset{\sim}{m}+1)}$

$$\hat{W} = W + P \sum_{j=0}^{m} S_j f_j(\phi)$$

$(P, \underbrace{S_0, \dots, S_m}_{\text{coord on } \check{\mathbb{P}}^m = \mathbb{P}(V_X^\vee)})$ — dual

P: section of a $\mathcal{L} \to \check{\mathbb{P}}^m$

by constr. the (ζ_e, ζ_ℓ) space of $\underline{\underline{\mathcal{T}_{\mathcal{X}}}}$ looks like:



$D(\mathcal{X})$ — universal hyperplane section of X

$\to \mathcal{X} \subset X \times \check{\mathbb{P}}^m$

$= \{(x,v) \in X \times \check{\mathbb{P}}^m \mid v(x) = 0\}$

$$\underline{\underline{D(\mathcal{X})}} = \langle \mathcal{C}, \overbrace{C_1, \dots, C_{K'}}\rangle$$

$$\mathcal{C} = D(\hat{Y}_{\zeta_\ell \ll -1}, \hat{W}_{\zeta_\ell \ll -1})$$

by definition $\mathcal{C}$ is Kuznetsov's HPD category of X and in some cases we can write $D(Z) = \mathcal{C}$ for some projective variety Z

Why HPD is interesting?

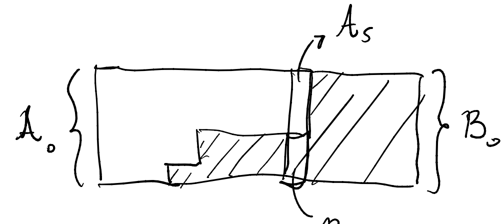
i) $D(Z) = \mathcal{C}$ has a dual $\underline{\text{SOD}}$ given $D(X) = \overbrace{\langle \mathcal{A}_0, \mathcal{A}_1(1), \dots, \mathcal{A}_m(m)\rangle}^{\text{S.O.D.}}$

$D(Z) = \langle \mathcal{B}_\ell(-\ell), \dots, \mathcal{B}_1(-1), \mathcal{B}_0\rangle$ (completely det. by $\mathcal{A}_0$ or $\mathcal{B}_0$)

$\underline{\mathcal{A}_0 \supset \mathcal{A}_1 \supset \dots \mathcal{A}_m}$ $\quad \mathcal{A}_i(i) = \mathcal{A}_i \otimes \mathcal{L}^{\otimes i}$

$\mathcal{B}_\ell \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$

s.t. $\underline{\mathcal{B}_0 \cong \mathcal{A}_0}$



$\langle \mathcal{A}_s, \mathcal{B}_s \rangle \cong \mathcal{A}_0 \cong \mathcal{B}_0$

ii) if $\underline{D(Z) = \mathcal{C}}$ we can take "linear sections"

$L \subset V_X^\vee \Rightarrow L^\perp \subset V_X$

L: linear subspace; $L^\perp$: orthogonal

$$\left.\begin{aligned} X_L &= X \cap L^\perp \\ Z_L &= Z \cap L \end{aligned}\right\} \text{ they are of the expected dimensions}$$

$$(\tilde{\mathcal{C}}_L =) D(Z_L) = \langle \mathcal{B}_\ell(n-\ell-1-r), \dots, \mathcal{B}_{n-r}(-1), \underline{\underline{\mathcal{C}_L}} \rangle \qquad r = \dim L$$

$$D(X_L) = \langle \underline{\underline{\mathcal{C}_L}}, \underbrace{\mathcal{A}_r(1), \dots, \mathcal{A}_k(k+1-r)}_{\text{in some cases all these categories are empty}} \rangle$$

and the categories $\tilde{\mathcal{C}}_L = D(Z_L)$ and $D(X_L)$ also have a GLSM construction, starting from $\mathcal{T}_X$.