A new look at integrable σ-models and their deformations

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Based on several papers of the speaker,

including arXiv:2005.01812 (with D. Lüst), arXiv:2006.14124 and an upcoming one

PLAN

– Flag manifold σ-models from spin chains –

– Gauged linear σ-models $\to$ gauged chiral bosonic Gross-Neveu systems –

– Ricci flow: the explicit universal solution –

– Chiral anomalies, inclusion of fermions, SUSY –

$$\mathscr{F}_{n_1,\ldots,n_S} = \frac{SU(N)}{S(U(n_1)\times\cdots\times U(n_S))}, \qquad \sum_{i=1}^{S} n_i = N$$

Complex structure $\leftrightarrow$ ordering of $n_1,\ldots,n_S$.

Complex definition: $\mathscr{F} = GL(N,\mathbb{C})/P$

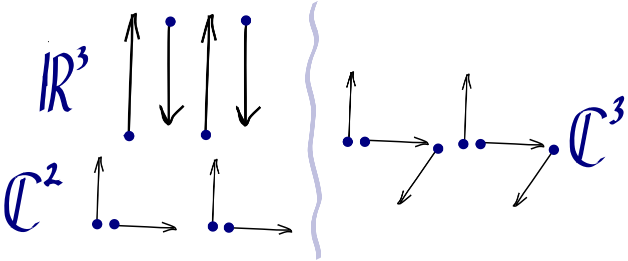
P = parabolic subgroup stabilizing the flag

$$0 \in L_1 \subset \ldots \subset L_S = \mathbb{C}^N, \qquad \dim_{\mathbb{C}} L_k := d_k = \sum_{i=1}^{k} n_i\,.$$

(Can be generalized to other groups)

Flag manifolds.

They arise as effective continuum theories of spin chains with $SU(N)$-symmetry. ($SU(2)$-case: [Haldane '83], [Affleck '85])
The idea is that the flag manifold is the space of Néel vacua of the classical chain:



– Geometric theory: [DB '11-'12]
– Analysis of spin chains: [Affleck et.al. '17 ($SU(3)$), '19, '20 ($SU(N)$)]
– Discrete 't Hooft anomalies [Tanizaki & Sulejmanpasic '18, Seiberg et.al. '18]
predict gapless spectrum ↔ match with rigorous spin chain results
[Lieb, Schultz, Mattis '61, Affleck, Lieb '86]

The σ-model in conventional terms.

In this talk I will discuss mostly the flag σ-models that are (conjecturally) integrable. Symmetric space examples date back to
[Pohlmeyer '76, Zakharov-Mikhaylov '78, Eichenherr-Forger '79]

In general, take the flag manifold $\frac{G}{H}$, define $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $J = -g^{-1}dg = J_{\mathfrak{h}} + J_{\mathfrak{m}}$

Metric G: the 'Killing metric' $ds^2 = \mathrm{Tr}(J_{\mathfrak{m}} J_{\mathfrak{m}})$
Complex structure J
B-field: fundamental Hermitian form of the metric $B = \mathsf{G} \circ \mathsf{J}$

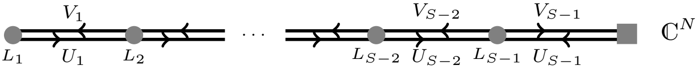
Action: $\mathcal{S}[\mathsf{G}, \mathsf{J}] := \int_\Sigma d^2z\, \mathsf{G}_{m\overline{n}}\, \partial U^m \overline{\partial U^n}$ [DB, 2014]

Note: metric G is only Kähler iff $\frac{G}{H}$ is symmetric (= Grassmannian)
Geodesics of G are homogeneous [Alekseevsky & Arvanitoyeorgos '07].

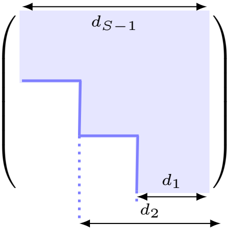
Kähler case: GLSM $\leftrightarrow$ Kähler quotients.
Grassmannian: $\mathsf{Gr}(m, N) = \mathrm{Hom}(\mathbb{C}^m, \mathbb{C}^N)//U(m)$. Lagrangian

$$\mathscr{L} = \mathrm{Tr}((\overline{D}U)^\dagger(\overline{D}U)), \qquad \overline{D}U := \overline{\partial}U - i\,U\,\overline{\mathcal{A}}\,, \qquad U^\dagger U = \mathbb{1}_m\,.$$

[Cremmer, Scherk '78, D'Adda, Lüscher, di Vecchia '78].
Flag manifold with Kähler metric: GLSM $\leftrightarrow$ Nakajima (quiver) varieties
[Nakajima '94, Nitta '03, Donagi & Sharpe '08].



Flag manifold with 'Killing metric' (not Kähler for $S > 2$): a 'gauge field' [DB '17]

$\mathcal{A} =$ 

, $\overline{\mathcal{A}} = (\mathcal{A})^\dagger$,

A 'reduced' gauge field!
I will explain the meaning of this later on in the talk.

Same models may be obtained by the approach of [Costello-Yamazaki '2019] from a coupling of two $\beta\gamma$-systems. Together with the GLSM approach this leads to the following alternative definition [DB '20]

$$\mathscr{L} = \overline{\Psi_a} \not{D} \Psi_a + (r_s)_{ab}^{cd} \left(\overline{\Psi_a} \frac{1+\gamma_5}{2} \Psi_c \right) \left(\overline{\Psi_d} \frac{1-\gamma_5}{2} \Psi_b \right) ,$$

where

$$\Psi_a = \begin{pmatrix} U_a \\ \overline{V}_a \end{pmatrix} , \qquad a = 1, \ldots, N \qquad \text{'Dirac boson'}$$

r_s is the classical r-matrix of [Belavin, Drinfeld '80]:
$r_s = \frac{s}{1-s}\, \pi_+ + \frac{1}{1-s}\, \pi_- + \frac{1}{2} \frac{1+s}{1-s}\, \pi_0$ (solution of CYBE)

> σ-model = chiral gauged Gross-Neveu model (in bosonic incarnation)!

(Fermionic version: [Gross-Neveu '74, Witten '78])

Chiral symmetry ($\lambda \in \mathbb{C}^\times$) [Zumino, 1977; Mehta 1990] $U \to \lambda U, \quad V \to \lambda^{-1} V$ ensures that the interaction is quadratic in U and in V, so that we can integrate out V.

Massless Thirring model [Thirring '58], cf. [Swieca '77] for a review

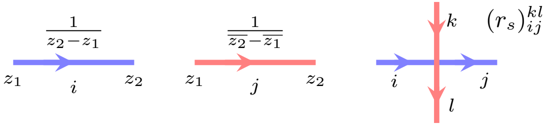
$N = 1$, ungauged: $\mathcal{A} = 0$, undeformed: $(r_s)_{ab}^{cd} = \delta_a^c\, \delta_b^d$

$\mathscr{L} = \overline{\Psi} \partial\!\!\!/ \Psi + \frac{1}{2} (\overline{\Psi} \gamma_\mu \Psi)^2 = V \bar{\partial} U - \overline{V} \partial \overline{U} + |U|^2 |V|^2$

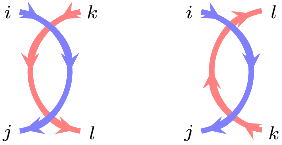
Integrate out V: $\quad \mathscr{L} = \frac{\bar{\partial} U\, \partial \overline{U}}{U \overline{U}} \quad \to \quad$ cylinder $\mathbb{C}^\times$ with multiplicative coordinate U

Vanishing β-function: bosonic Thirring vs. free boson (with a linear dilaton)

Feynman rules:



Diagrams contributing to the β-function at one loop:



β-function:

$$\beta_{ij}^{kl} = \sum_{p,q=1}^{N} \left((r_s)_{ip}^{kq} (r_s)_{pj}^{ql} - (r_s)_{ip}^{ql} (r_s)_{pj}^{kq} \right)$$

The Ricci flow equation $\dot{r}_{ij}^{kl} = \beta_{ij}^{kl}$ has a remarkably simple solution $s = e^{N\tau}$ (was conjectured in [Costello-Yamazaki 2019]).

Alternatively, return to the σ-model and solve the geometric Ricci flow equations

$$
\begin{aligned}
-\dot{g_{ij}} &= R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + 2\,\nabla_i \nabla_j \Phi\,, \\
-\dot{B_{ij}} &= -\frac{1}{2}\,\nabla^k\, H_{kij} + \nabla^k \Phi\, H_{kij}\,, \\
-\dot{\Phi} &= \text{const.} - \frac{1}{2}\,\nabla^k \nabla_k\, \Phi + \nabla^k \Phi \nabla_k \Phi + \frac{1}{24}\, H_{kmn} H^{kmn}
\end{aligned}
$$

$\mathbb{CP}^1$: the 'sausage' solution $s = e^{2\tau}$ $(N = 2)$ [Fateev, Onofri, Zamolodchikov, '1994]

$$ds^2 = \frac{(s^{-1} - s)|dW|^2}{(s + |W|^2)(s^{-1} + |W|^2)} \qquad \boxed{\text{Length} \sim |\log s|} \qquad 0 < s < 1$$

$\mathbb{CP}^{N-1}$: Ricci flow interpolates between a cylinder $(\mathbb{C}^\times)^{N-1}$ in the UV (asymptotic freedom) and a 'round' projective space of vanishing radius in the IR.

Anomalies [DB '20]

Remember $U \in \mathbb{C}^N$, so to pass to $\mathbb{CP}^{N-1}$ we need to gauge the chiral symmetry. However the symmetry is typically anomalous: recall Schwinger's effective action

$$\mathscr{S}_{\text{eff.}} = \frac{\xi}{2} \int dz\, d\overline{z}\, F_{z\overline{z}} \frac{1}{\triangle} F_{z\overline{z}}\,, \qquad F_{z\overline{z}} = i\,(\partial\overline{\mathcal{A}} - \overline{\partial}\mathcal{A}) \qquad \text{[Schwinger '1962]}$$

Not invariant under the complexified gauge transformations
$\mathcal{A} \to \mathcal{A} + \partial\alpha$, $\overline{\mathcal{A}} \to \overline{\mathcal{A}} + \overline{\partial}\overline{\alpha}$.

To cancel the anomaly one can add fermions minimally:

$$\mathscr{L} = \overline{\Psi_a} \not{D} \Psi_a + (r_s)_{ab}^{cd} \left(\overline{\Psi_a} \frac{1+\gamma_5}{2} \Psi_c \right) \left(\overline{\Psi_d} \frac{1-\gamma_5}{2} \Psi_b \right) + \overline{\Theta_a} \not{D} \Theta_a\,,$$
$$\Psi, \Theta \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2 \otimes \mathbb{C}^N)\,.$$

Incidentally these are the same fermions that cancel the anomaly in Lüscher's nonlocal charge [Abdalla et.al., 1981-84]!

Conjecture: all such flag manifold models with fermions are quantum integrable

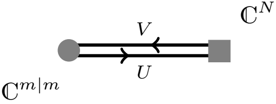
Super phase space Φ = complex symplectic (quiver) super-manifold
Matter fields U, V in reps. $\mathsf{W} \oplus \mathsf{W}^\vee$ of a complex gauge group G_{gauge}
(Global) action of a complex group G_{global}
Complex moment map μ for $G_{\text{global}} \circlearrowleft \Phi$

$$\mathscr{L} = \left(V \cdot \overline{\mathscr{D}} U + \overline{U} \cdot \mathscr{D} \overline{V} \right) + \kappa \, \mathrm{Tr}(\mu \, \overline{\mu})$$

Complex symplectic quotient instead of Kähler quotient!

Anomaly cancellation conditions: $\mathrm{Str}_{\mathsf{W}}(T_a) = 0, \quad \mathrm{Str}_{\mathsf{W}}(T_a T_b) = 0 \quad (T_a \in \mathfrak{g}_{\text{gauge}})$

Example. SUSY Grassmannian model $\mathsf{Gr}(m, N)$, $G_{\text{global}} = \mathfrak{sl}_N(\mathbb{C})$



$$G_{\text{gauge}} = \left\{ g = \begin{pmatrix} \lambda & 0 \\ \chi & \lambda \end{pmatrix} \in SL(m|m), \; \lambda \in GL(m, \mathbb{C}) \right\}$$

$U \in \mathrm{Hom}(\mathbb{C}^{m|m}, \mathbb{C}^N)$ and $V \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^{m|m})$.

- Integrable sigma-models beyond symmetric target spaces [DB '14$^+$, Costello-Yamazaki 2019]
- GLSM formulation beyond Kähler target spaces [DB '17]
- σ-models = gauged chiral Gross-Neveu models [DB '20]
- The one-loop β-function is universal for all of these models (one-loop exact?)
- (Complicated) Ricci flow eqs. have a simple (ancient) solution, interpolating between the homogeneous metric and a cylinder
- Anomalies $\to$ Super phase spaces with anomaly cancellation conditions ['20, to appear]
- Interactions are polynomial $\to$ relation to Ashtekar variables of GR (direct derivation for $\frac{SL(2,\mathbb{R})}{SO(2)}$ [Brodbeck & Zagermann '00])
- Poisson brackets of Lax operators are ultralocal [Delduc et.al. '19]
- Should be possible to construct the full quantum theory: S/R-matrix, spectrum (thermodynamic Bethe ansatz), analyze resurgence, etc.
  Possibly using the ODE/IQFT approach
  [Bazhanov, Lukyanov, Zamolodchikov 98$^+$, Bazhanov, Kotousov, Lukyanov '17]
- $\cdots$

Let us thank Eric and his co-organizers
Cyril, Johanna and Ilarion
for setting up this nice conference,
which ran very smoothly!

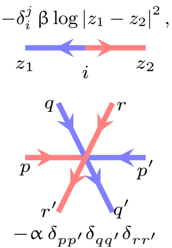
We appreciate the effort
you have invested in it!

Instead of working in a standard gauge like $\overline{U}U = 1$, we can choose

$$U_N = 1 \qquad \text{(inhomogeneous gauge)}$$

Varying the action w.r.t. $\overline{\mathcal{A}}$, we get $UV = 0$, i.e. $V_N = -\sum_{j=1}^{N-1} U_j V_j$.
Substituting in the Lagrangian, we get

$$\mathscr{L} = \sum_{k=1}^{N-1} \left(V_k \overline{\partial} U_k - \overline{V}_k \partial \overline{U}_k + \beta |V_k|^2\right) +$$
$$+ \underbrace{\sum_{l,m=1}^{N-1} a_{lm} \, |U_l|^2 |V_m|^2 + \gamma \, \Big|\sum_{p=1}^{N-1} U_p V_p\Big|^2}_{\text{quartic vertices}} + \underbrace{\alpha \left(\sum_{k=1}^{N-1} |U_k|^2\right) \Big|\sum_{p=1}^{N-1} U_p V_p\Big|^2}_{\text{sextic vertices}}$$



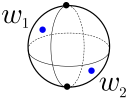
Instead of a σ-model we obtained a theory
with polynomial interactions! Parallel with Ashtekar variables:

- Direct derivation for $\frac{SL(2,\mathbb{R})}{SO(2)}$ [Brodbeck & Zagermann '00]
- Interactions are polynomial
- Degenerations are allowed (compare with nilpotent orbits)

A semi-holomorphic 4D Chern-Simons theory on $\Sigma \times \mathscr{C}$, where Σ = 'topological plane' $(z, \overline{z}) \to$ worldsheet to-be
$\mathscr{C}$ = complex curve $(w, \overline{w})$ with a holomorphic differential $\omega = dw \neq 0$.
$K_{\mathscr{C}} = 0$ implies $\mathscr{C} \simeq \mathbb{C}, \mathbb{C}^*, E_\tau$. The action:

$$S_{\mathrm{CS}} = \frac{1}{\hbar} \int\limits_{\Sigma \times \mathscr{C}} \omega \wedge \mathrm{Tr}\left(A \wedge (dA + \frac{2}{3} A \wedge A) \right),$$

where $A = A_z dz + A_{\overline{z}} d\overline{z} + A_{\overline{w}} d\overline{w}$. One couples this theory to two $\beta\gamma$ systems, with target space $T^*\mathcal{M}$, where $\mathcal{M}$ is a complex homogeneous space:



$\mathscr{C} \approx \mathbb{C}^*$

$$S_{def} = \int\limits_{\Sigma} d^2 z \, (p_i D_{\overline{z}}^{(w_1)} q^i + \overline{p}_i D_z^{(w_2)} \overline{q}^i),$$

where $\quad D_{\overline{z}}^{(w_1)} q^i = \partial_{\overline{z}} q^i - \sum_a (A_{\overline{z}}^{(w_1)})_a v_a^i$.

'Light-cone' gauge $A_{\overline{w}} = 0$, solve for $A_z, A_{\overline{z}}$. In this gauge the equations are

$$\begin{aligned} &\partial_{\overline{z}} A_z - \partial_z A_{\overline{z}} + [A_z, A_{\overline{z}}] = 0, \\ &\partial_{\overline{w}} A_z = \delta^{(2)}(w - w_1) \sum_a p_i v_a^i \, \tau_a \\ &\partial_{\overline{w}} A_{\overline{z}} = \delta^{(2)}(w - w_2) \sum_a \overline{p}_{\overline{i}} v_a^{\overline{i}} \, \tau_a \, . \end{aligned}$$

Family $A_z(w), A_{\overline{z}}(w)$ of flat connections, depending meromorphically on w!

Key observation: Green's function $\overline{\partial}_{\overline{w}}^{-1}$ = classical r-matrix [Belavin, Drinfeld '80]

Rational case: $r(w) = \frac{\sum \tau_a \otimes \tau_a}{w} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r(w) = \frac{\mathbb{1}}{w} \in \mathrm{End}(\mathfrak{g})$.

Trigonometric case: let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ (complex structure on G, Manin triple, etc.), then $r_{\mathrm{comp.}}(u) = \frac{\sum \tau_a^+ \otimes \tau_a^-}{1-u} - \frac{\sum \tau_a^- \otimes \tau_a^+}{1-u^{-1}} \in \mathfrak{g} \otimes \mathfrak{g}$, i.e. $r_{\mathrm{comp.}}(z) = \frac{\Pi_+}{1-u} - \frac{\Pi_-}{1-u^{-1}} \in \mathrm{End}(\mathfrak{g})$.

Upon integrating out $A_z, A_{\overline{z}}$, we get (rational case)

$$S = \int d^2z \left(p_i \partial_{\overline{z}} q^i + \overline{p}_i \partial_z \overline{q}^i + \mathbf{r}_{w_1 - w_2}\left(p_i v_a^i\, \tau_a, \overline{p}_{\overline{i}} v_a^{\overline{i}}\, \tau_a\right)\right) =$$

$$= \int d^2z \left(p_i \partial_{\overline{z}} q^i + \overline{p}_i \partial_z \overline{q}^i + \frac{1}{w_1 - w_2} \sum |p_i v_a^i|^2\right) =$$

$$= \text{integrate out} \quad p, \overline{p} \quad (\text{the fiber of} \quad T^*\mathcal{M}) \sim$$

$$\sim \int d^2z \left(G_{i\overline{j}} \partial_{\overline{z}} q^i \partial_z \overline{q}^{\overline{j}}\right) \qquad \text{with} \qquad G_{i\overline{j}} = \left(\sum_a v_a^i v_a^{\overline{j}}\right)^{-1}$$

Invertibility $\leftrightarrow$ Homogeneous space

Rational case: the flag manifold σ-model described earlier. [DB, 2019]

But this also provides deformations of those models, trigonometric and elliptic. We pass over to this topic, starting with deformations of the $\mathbb{CP}^{n-1}$ model.

The conjecture of integrability of the models is based on the following evidence:

- The zero-curvature representation

$$A_u = \frac{1+u}{2} K_z dz + \frac{1+u^{-1}}{2} K_{\overline{z}} d\overline{z}, \qquad u \in \mathbb{C}^*\,,$$

  where K = Noether current (*flat*).
- Involutivity of the integrals of motion [Delduc et. al. '19]
- Explicit classical solutions $\left(\frac{U(3)}{U(1)^3}\right)$ [DB '16], generalizing [Din, Zakrzewski '80]
- Analogy with the case of symmetric spaces (review: [Zarembo '17])

Symmetric spaces of $SU(N)$: Grassmannians $G(m, N) := \frac{SU(N)}{S(U(m)\times U(N-m))}$

Consider the Ricci flow for the metric

$$-\dot{g_{ij}} = R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g^{mm'} g^{nn'} + \nabla_i \mathcal{D}_j \Phi + \nabla_j \mathcal{D}_i \Phi \,, \qquad \mathcal{D}\Phi = d\Phi - \mathcal{E} \,.$$

The structure of the blow-up (strong coupling) is as follows: $g_{ij} \sim (1-s)\,(g_{\text{hom.}})_{ij}$.

For the homogeneous metric $\mathcal{D}_i \Phi = 0$. Since $s = e^{N\tau}$, in the limit $-\dot{g_{ij}} \to N\,(g_{\text{hom.}})_{ij}$
Homogeneous (Killing) metric $g_{\text{hom.}}$ satisfies a generalized Einstein condition

$$R_{ij} + \frac{1}{4} H_{imn} H_{jm'n'} g_{\text{hom.}}^{mm'} g_{\text{hom.}}^{nn'} = N\,(g_{\text{hom.}})_{ij} \,.$$

For Grassmannians $H = 0$, i.e. $R_{ij} = N\,(g_{\text{hom.}})_{ij}$.
N= first Chern number of the tangent bundle
$c_1(G(m,N)) = N\,[\mathscr{C}]$ ($\mathscr{C}$=generator) – independent of m!

β-function for a symmetric space = dual Coxeter number (independent of H in $\frac{G}{H}$).
We observe this for non-symmetric spaces – can be proven directly!

Let $g \in SU(N)$, $g^{-1}dg = i\,(\Phi\, dz + \overline{\Phi}\, d\overline{z})$.
$\overline{\partial}\Phi + i\,[\overline{\Phi}, \Phi] = 0\,.$ Flatness of Noether current, Principal Chiral Model
Impose the condition $\Phi^S = 0$ (closure of nilpotent orbit in $\mathfrak{gl}_N$)

Assume

$$\Phi^S = 0, \qquad \text{and} \qquad \Phi^{S-1} \neq 0\,.$$

The map $\Phi(z, \overline{z})$ satisfying the E.O.M. defines a flag

$$0 \subset \mathrm{Ker}(\Phi) \subset \mathrm{Ker}(\Phi^2) \subset \cdots \subset \mathrm{Ker}(\Phi^S) \simeq \mathbb{C}^N$$

This is a point in

$$\mathscr{F} := \frac{U(N)}{U(\kappa_1) \times \cdots \times U(\kappa_S)}, \qquad \text{where}$$

$$\kappa_j = \dim \mathrm{Ker}(\Phi^j)/\mathrm{Ker}(\Phi^{j-1}) \quad = \text{number of Jordan blocks of size at least } i\,.$$

Claim: the map $\Sigma \to \mathscr{F}$ is a solution of a flag manifold sigma-model
This is a map to a single orbit: type does not change due to e.o.m. [DB, 2019]
I will now describe the model.

Simplest deformation: 2D target space with a $U(1)$-isometry [DB-Lüst, 2020]

$$ds^2 = \sum_{i,j=1}^{2} G_{ij}\, dX^i dX^j = \frac{1}{4\, g(\mu)}\, d\mu^2 + g(\mu)\, d\phi^2 \, .$$

For what $g(\mu)$ is the model integrable?
Mechanical reduction always integrable (2 integrals of motion).
Generalized Pohlmeyer map: set $G_{ij}\, \partial X^i \overline{\partial} X^j = \cosh\chi$.

Sinh-Gordon equation replaced by $\quad \overline{\partial}\partial\chi - 2g''(\mu)\, \sinh\chi = 0\, .$
Add the equation for μ: $\quad \partial\overline{\partial}\mu - 2\, g'(\mu)\, \cosh\chi = 0\, .$

The two eqs. follow from a single Lagrangian if $g(\mu) = b + a\, \cosh\mu$:

$$\mathscr{L} = \frac{1}{2}\partial\mu\overline{\partial}\mu + \frac{1}{2}\partial\chi\overline{\partial}\chi + 2a\, \cosh\mu\, \cosh\chi\ =$$
$$= \left(\frac{1}{2}\partial\widetilde{\mu}\, \overline{\partial}\widetilde{\mu} + a\, \cosh\left(\sqrt{2}\widetilde{\mu}\right)\right) + \left(\frac{1}{2}\partial\widetilde{\chi}\, \overline{\partial}\widetilde{\chi} + a\, \cosh\left(\sqrt{2}\widetilde{\chi}\right)\right)$$

Ricci flow in the Kähler case.

The (Kähler) metric of the 'sausage' [Fateev, Onofri, Zamolodchikov, 1994]

$$ds^2 = \left(\frac{1}{s} - s\right) \frac{|dW|^2}{(s + |W|^2)(\frac{1}{s} + |W|^2)}, \qquad 0 < s < 1\,.$$

$\mathbb{CP}^{N-1}$ also has a generalized Kähler deformation [Demulder et.al. 2020], constructed along the lines of [Delduc, Magro, Vicedo 2013]. The B-field has the form $B = \sum_i b_i \wedge d\phi_i$, so T-dualizing all angles we get rid of it.
T-dual geometry is Kähler with potential [DB-Lüst, 2020]

$$\mathcal{K} = \sum_{j=1}^{N} (Z_j \overline{Z}_{j-1} - \overline{Z}_j Z_{j-1}) + 2 \sum_{j=1}^{N} P(t_j - t_{j-1} - 2\,\tau), \qquad t_j = Z_j + \overline{Z_j}\,,$$

where $P(t) = \mathrm{Li}_2(e^{-t}) + \frac{t^2}{4}$.
$\mathcal{K}$ not invariant under $Z_i \to Z_i + \delta\alpha_i$. T-duality does not preserve the Kähler property (otherwise T: chiral $\leftrightarrow$ twisted chiral [Rocek, Verlinde 1991]).

The metric satisfies the (simple) Ricci flow equation $-\frac{dg_{i\bar{j}}}{d\tau} = 4\,R_{i\bar{j}}$ with $s = e^{N\tau}$.