Worldsheet instantons in Heterotic string theory

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GLSM 2020
08/21/2020

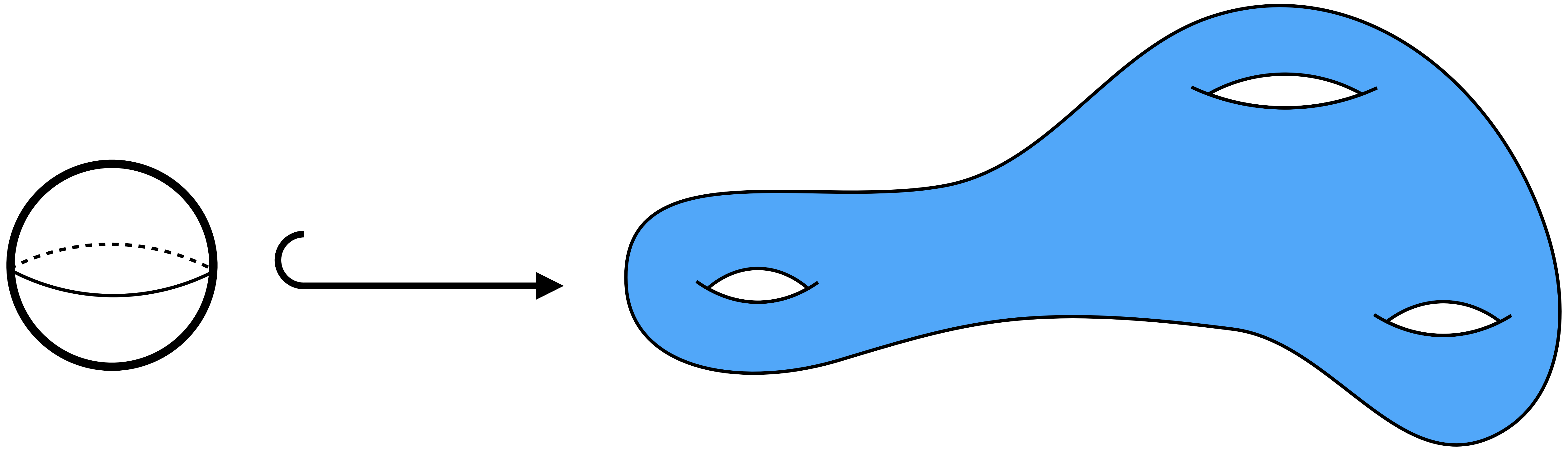
Based on:

[Buchbinder, Lukas, Ovrut, FR, 1912.07222]

[Buchbinder, Lukas, Ovrut, FR, 1912.08358]

[Buchbinder, Lukas, Ovrut, FR, 1707.07214]

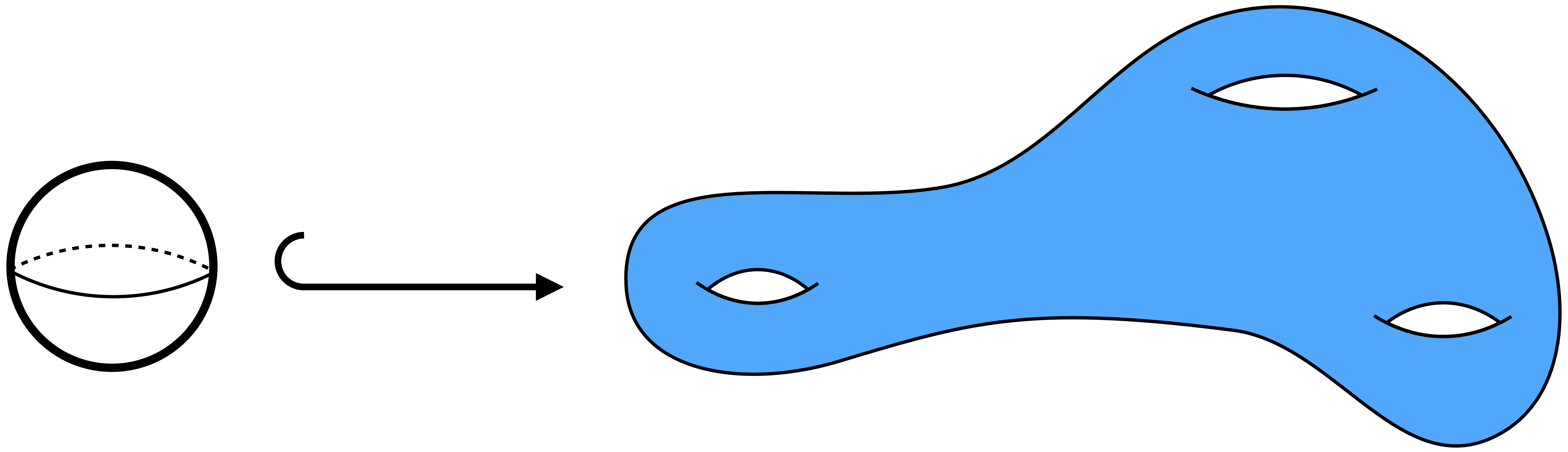
Motivation



- Contributions depend on
  - Kahler moduli t_i
  - CS moduli κ_I
  - Bundle moduli α_a

$$W_\gamma = \sum_{i=1}^{n_\gamma} \frac{\mathrm{Pfaff}(\overline{\partial}_{V_{\gamma_i}\otimes\mathcal{O}_{\gamma_i}(-1)})}{[\det(\overline{\partial}_{\mathcal{O}_{\gamma_i}})]^2 \det(\overline{\partial}_{N\gamma_i})} \exp\left[-\int_\gamma \frac{J}{2\pi\alpha'} - iB\right]$$

Motivation



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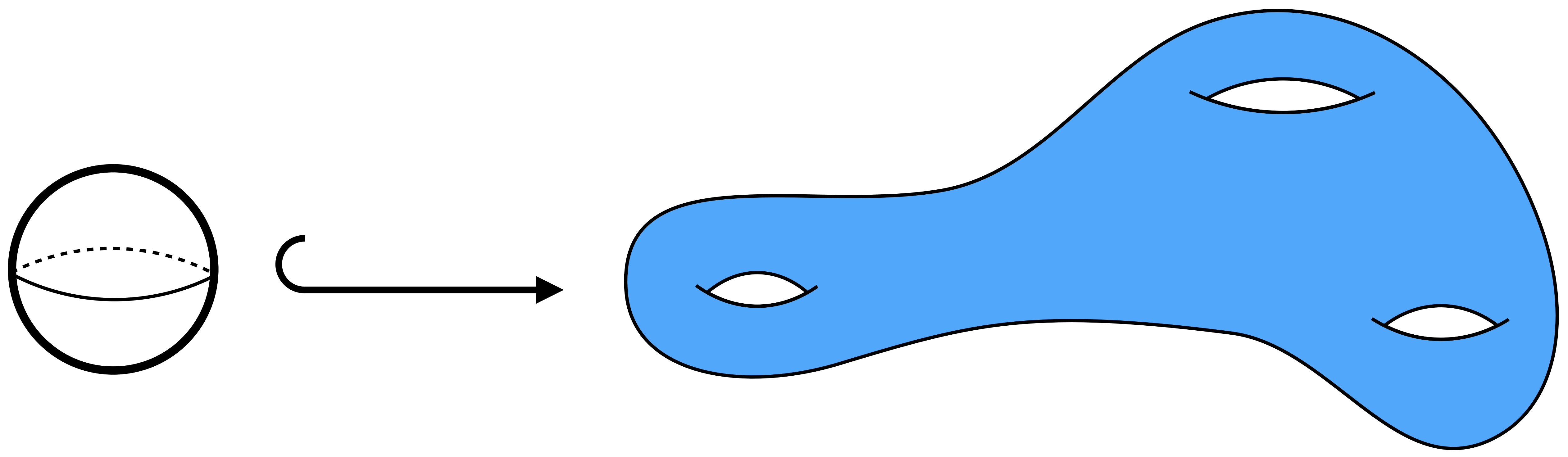
Genus 0 GW invariants

Motivation



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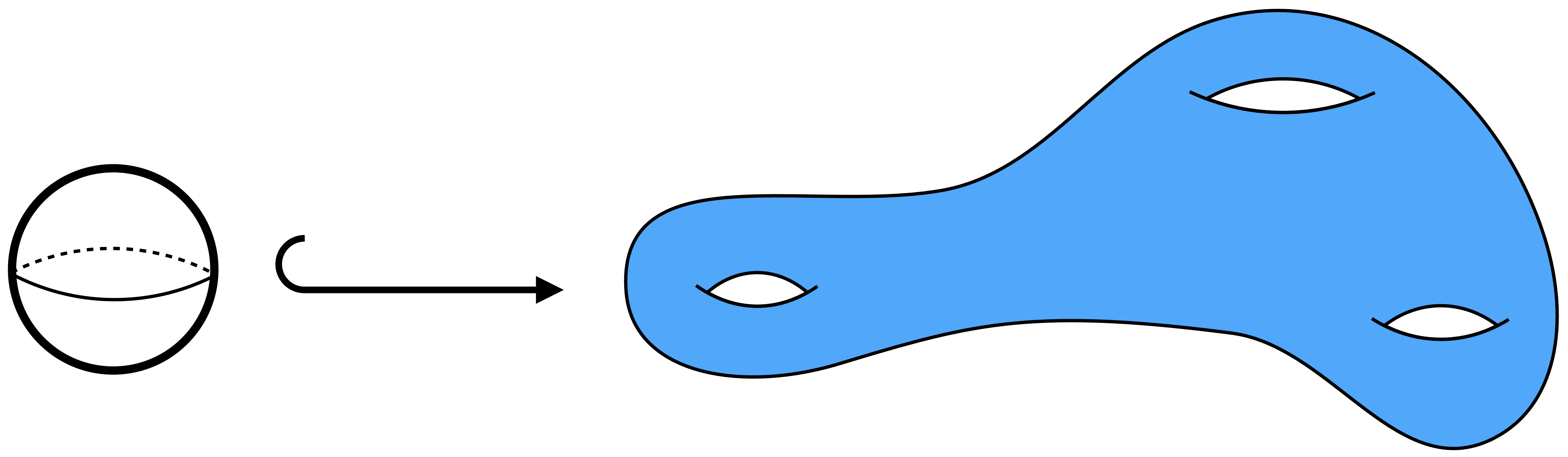
Integral over right-moving fermionic WS d.o.f.

Motivation



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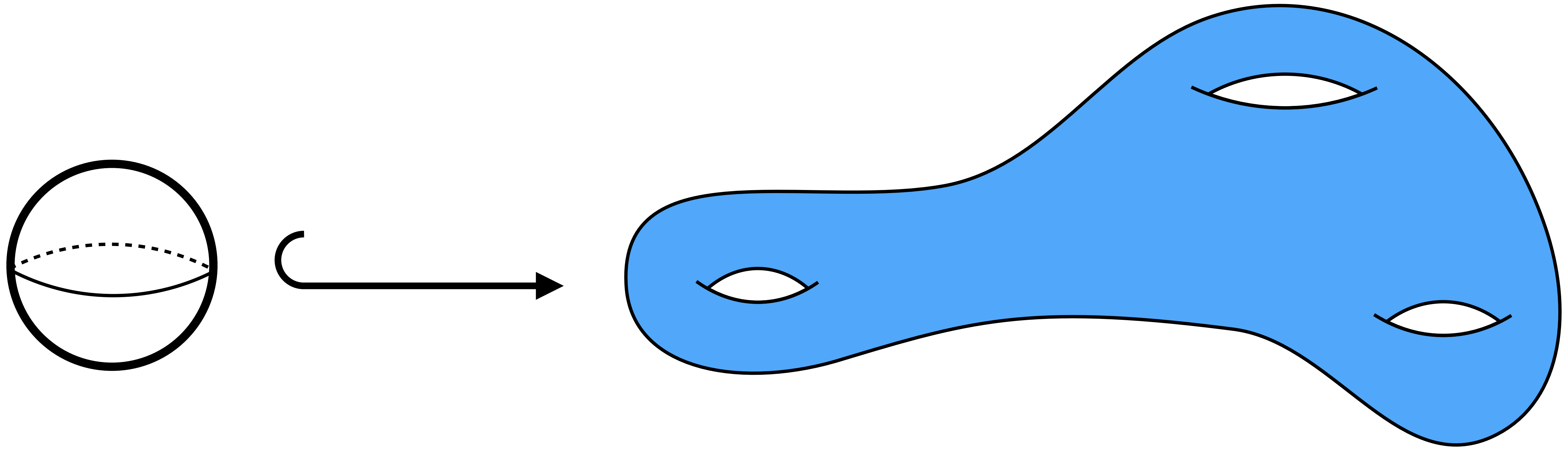
Vector bundle restricted to curve $\gamma = \mathbb{P}^1$

Motivation



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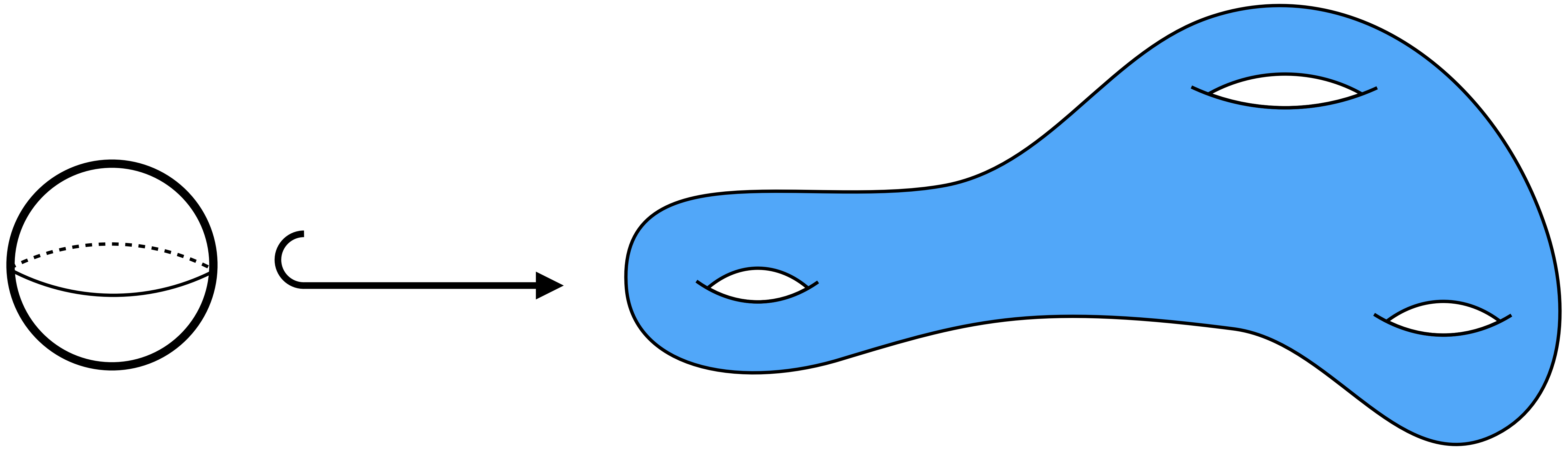
Spin bundle on $\gamma = \mathbb{P}^1$

Motivation



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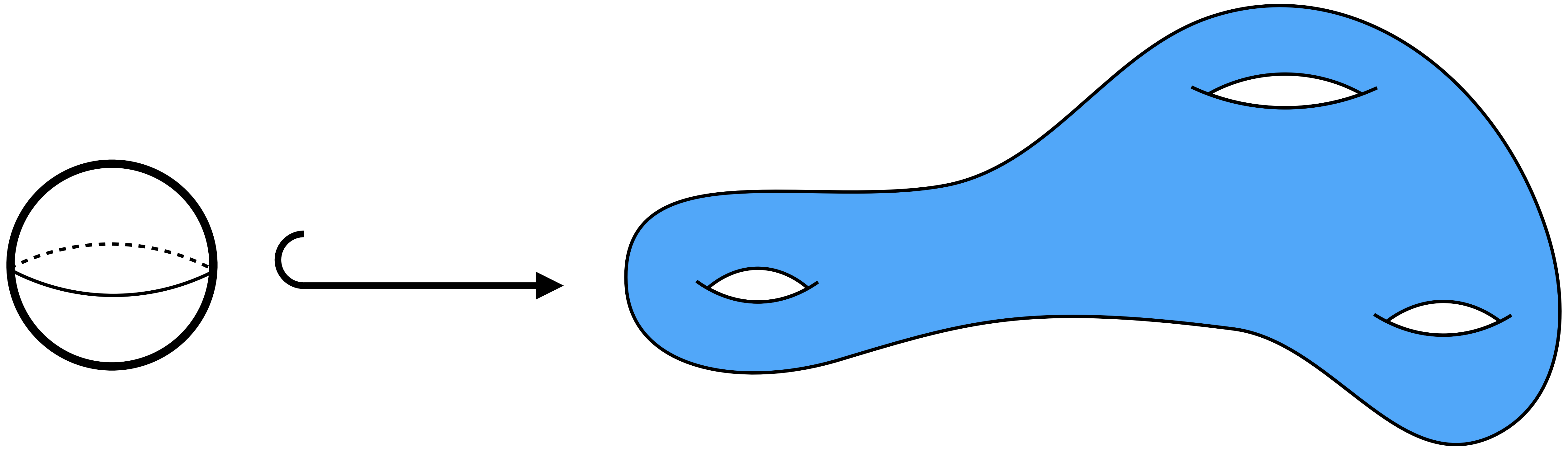
Introduce notation $V_{-1} := V \otimes \mathcal{O}_\gamma(-1)$

Motivation



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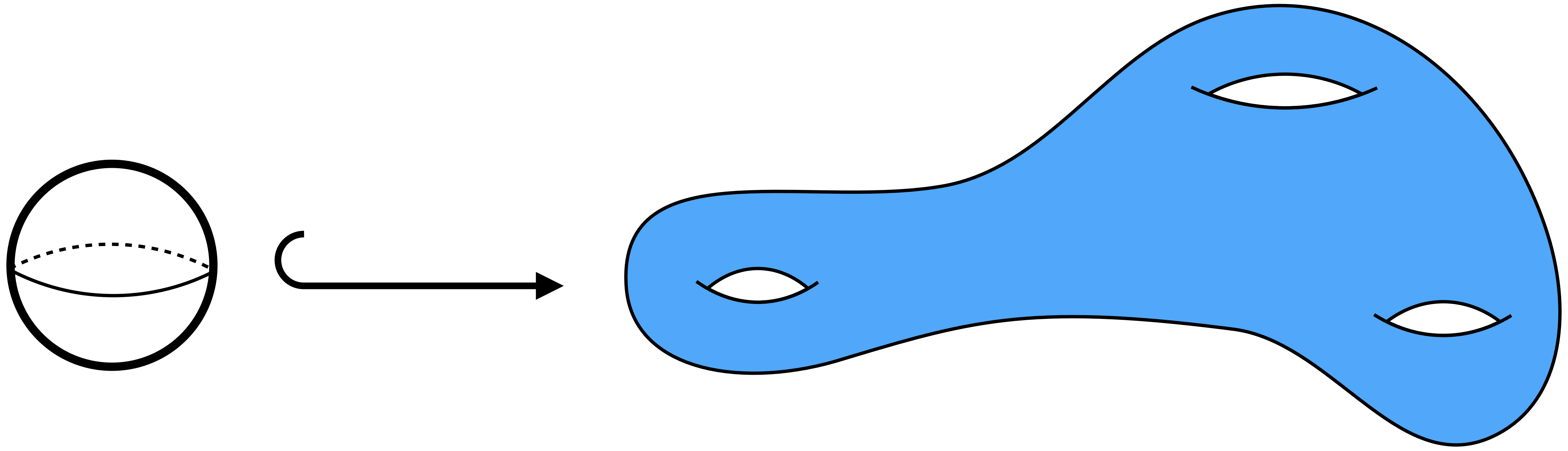
Constant

Motivation



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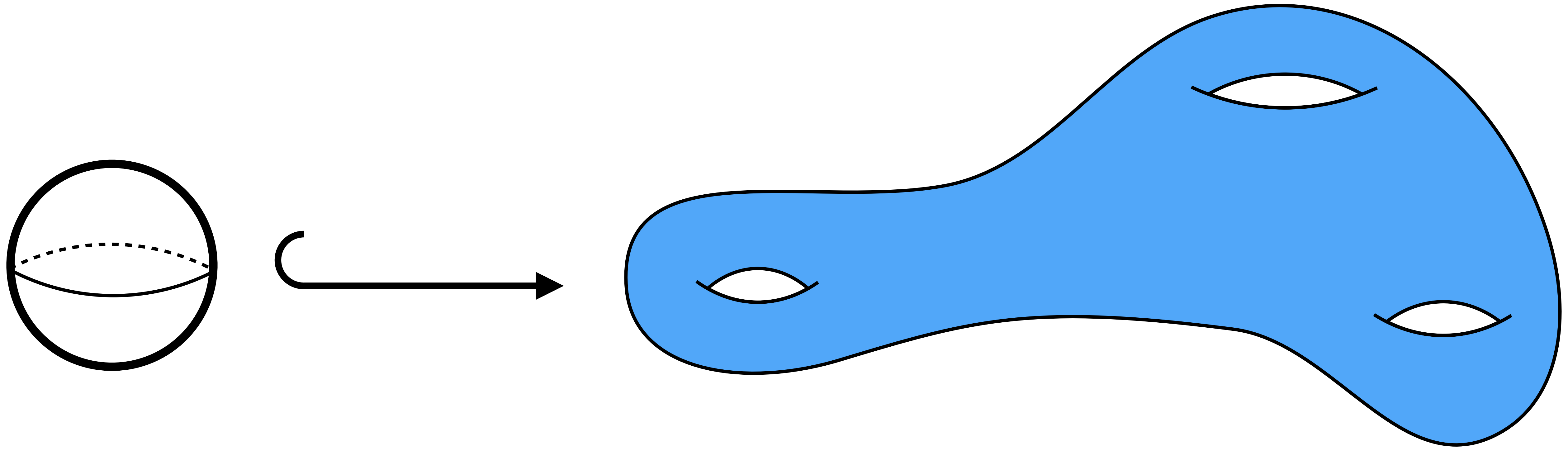
Integral over bos d.o.f.

Note: $N\mathbb{P}^1 = \mathcal{O}(-1) \times \mathcal{O}(-1)$

Motivation



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Complexified Kahler form

Motivation

- Work at arbitrary but fixed CS $\Rightarrow$ Focus on computing the Pfaffians
- $\mathrm{Pfaff} = 0 \quad \Leftrightarrow \quad \overline{\partial}_{V_{-1}}$ has a zero-mode $\quad \Leftrightarrow \quad h^0(\mathbb{P}^1, V_{-1}) \neq 0$
- $h^0(\mathbb{P}^1, V_{-1}) = 0$ generically, but cohomology can jump at codim 1 locus
- Pfaff is proportional to poly that describes the jump locus

[Buchbinder, Donagi, Ovrut, hep-th/0205190]

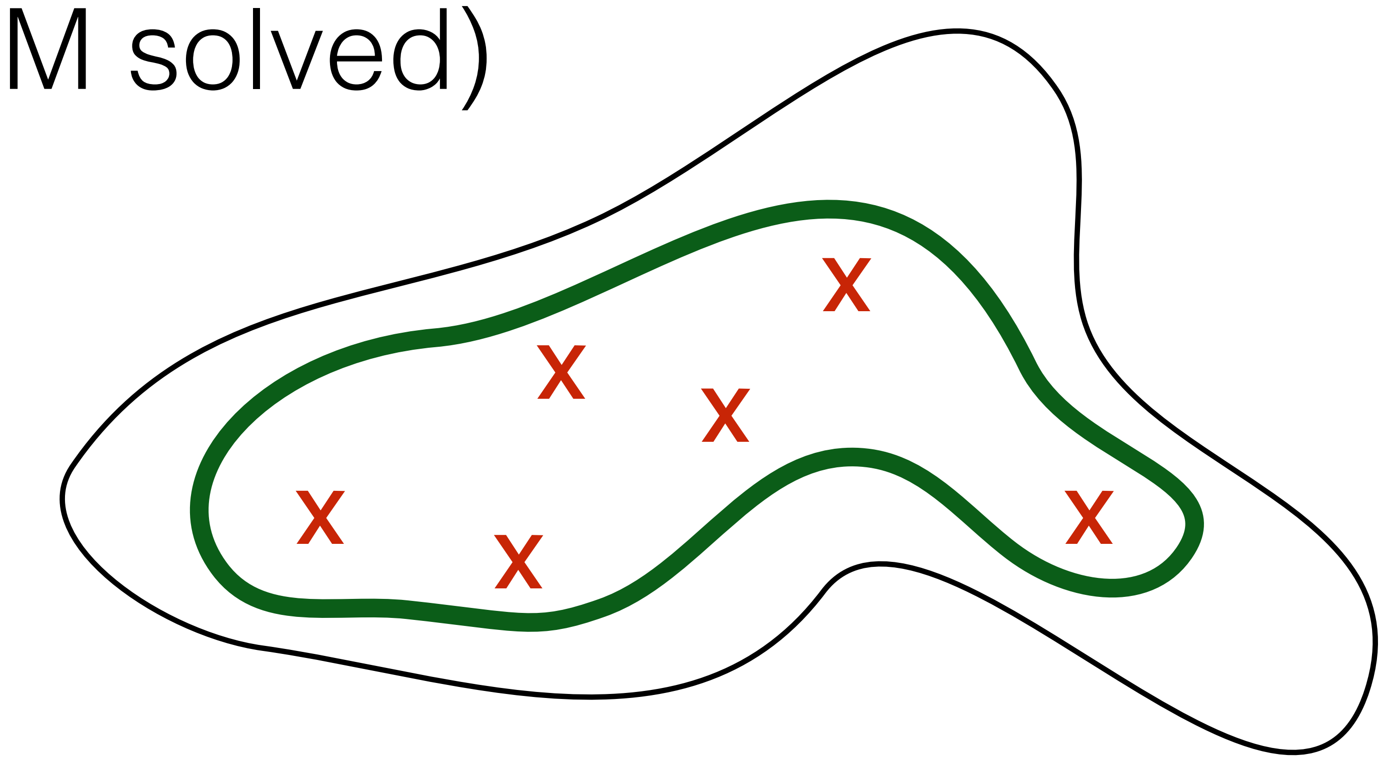
- To compute $\mathrm{Pfaff}(\kappa_I, \alpha_a)$:

1. Find the coordinates $\vec{z}_i^{\,*}$ of all representatives γ_i of $[\gamma] = [\mathbb{P}^1]$
2. Parameterize bundle moduli space
3. Compute the Pfaffians: $\sum_{i=1}^{n_\gamma} \mathrm{Pfaff}_{\gamma_i}(\kappa_I, \alpha_a) = \sum_{i=1}^{n_\gamma} \lambda_i \, p_{r\,;\,d_1,\ldots,d_{h^{11}}}\,(\alpha_a\,;\,\vec{z}_i^{\,*}(\kappa_I))$

$\in \mathbb{C}^*$

Motivation

- Beasley-Witten cancellation: **[Beasley, Witten hep-th/0304115]**
  - Consistent compactification (stable vector bundle, BI+HYM solved)
  - Gauge bundle pulls back from the ambient space
  - Right-moving WS SUSY
  - Instanton moduli space compact



- $$\sum_{i=1}^{n_\gamma} \mathrm{Pfaff}_{\gamma_i}(\kappa_I, \alpha_a) = \sum_{i=1}^{n_\gamma} \lambda_i \, p_{r\,;\,d_1,\ldots,d_h^{11}} \left(\alpha_a\,;\,\vec{z}_i^{\,*}(\kappa_I)\right) = 0$$
- Cancellation has to hold at any point in moduli space
  - If $p_{r\,;\,d_1,\ldots,d_h^{11}} \left(\alpha_a\,;\,\vec{z}_i^{\,*}(\kappa_I)\right)$ are linear independent ⇒ no BW cancellation
  - If $p_{r\,;\,d_1,\ldots,d_h^{11}} \left(\alpha_a\,;\,\vec{z}_i^{\,*}(\kappa_I)\right)$ are linear dependent ⇒ thousands of terms can cancel by choosing $n_\gamma = \mathcal{O}(20)$ complex constants λ_i (highly non-trivial if there is not a reason why the monomials align this way)

Motivation

- Checked for quintic $(h^{11} = 1,\ n_\gamma = 2875)$ in (half-) linear sigma model **[Beasley, Witten hep-th/0304115]**
- Compactness of instanton moduli space very hard to check
- Simple criterion for GLSMs **[Bertolini, Plesser 1410.4541]**

	$\mathcal{Z}_I$	P_k	A_α	B_β	S	Ξ
$\mathrm{U}(1)^i$	Q_I^i	$-q_k^i$	a_α^i	$-b_\beta^i$	Q_S	Q_Ξ
$\mathrm{U}(1)_L$	0	0	-1	1	1	-1
$\mathrm{U}(1)_R$	0	1	0	1	1	0
Interpretation	Geometry coordinates	Geometry constraints	Monad A-terms	Monad B-terms	Spectator	Spectator

- Zero-modes of s and b can make instanton moduli space non-compact
- For $\gamma = \sum_r w_r \mu_r$ check $h^0(\mathcal{O}_\gamma(-b^i_\beta \cdot w_i - 1)) = 0\,, \quad h^0(\mathcal{O}_\gamma(Q^i_s \cdot w_i - 1)) = 0$

Outline

- Curves in Complete Intersection Calabi-Yaus
- Compute the Pfaffian
  - Monad bundles
  - Extension bundles
- Comparing GLSM and alg. geom. conditions
  - Vanishing instanton sum and Hilbert functions
  - Pfaffians and GLSM quantities

Curves in Complete Intersection CYs

CICY

[Candelas, Dale, Lutken, Schimmrigk `88; Green and Hubsch `88]

$$\left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^4 & 2 & 1 & 2 \end{array}\right]$$

	$\mathcal{Z}_{1,2}$	$\mathcal{Z}_{3,4}$	$\mathcal{Z}_{5,6,7,8,9}$	P_1	P_2	P_3	
$U(1)_1$	1	0	0	-1	-1	0	$n_\gamma = 16$
$U(1)_2$	0	1	0	-2	0	0	$n_\gamma = 40$
$U(1)_3$	0	0	1	-2	-1	-2	$n_\gamma = 112$

[Hosono, Klemm, Theisen, Yau hep-th/9308122]

- Focus on CICYs with ambient space $\mathbb{P}^1$ factors
- Two possibilities
  - Charges of two P_i are 1, rest zero (type I)
  - Charges of one P_i is 2, rest zero (type II)

CICY

$$\left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^4 & 2 & 1 & 2 \end{array}\right]$$

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$\kappa_0 z_1 z_3^2 z_5^2 + \kappa_1 z_1 z_3 z_4 z_5^2 + \ldots = 0$

$\tilde{\kappa}_0 z_1 z_5 + \tilde{\kappa}_1 z_1 z_6 + \ldots = 0$

$\tilde{\tilde{\kappa}}_0 z_5^2 + \tilde{\tilde{\kappa}}_1 z_5 z_6 + \ldots = 0$

$$\Downarrow \qquad\qquad \Downarrow$$

$z_1 p_{22}(z_{3,4};\, z_{5,\ldots,9}) + z_2 q_{22}(z_{3,4};\, z_{5,\ldots,9}) = 0$

$z_1 r_{01}(z_{3,4};\, z_{5,\ldots,9}) + z_2 s_{01}(z_{3,4};\, z_{5,\ldots,9}) = 0$

$$\Downarrow$$

$p_{22}(z_{3,4};\, z_{5,\ldots,9}) = q_{22}(z_{3,4};\, z_{5,\ldots,9}) = r_{01}(z_{3,4};\, z_{5,\ldots,9}) = s_{01}(z_{3,4};\, z_{5,\ldots,9}) = 0 :$ z_1, z_2 arbitrary $\rightarrow \mathbb{P}^1$

CICY

$$\rightarrow \left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^4 & 2 & 1 & 2 \end{array}\right]$$

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$n_\gamma = 16$
$n_\gamma = 40$
$n_\gamma = 112$

$\kappa_0 z_1 z_3^2 z_5^2 + \kappa_1 z_1 z_3 z_4 z_5^2 + \ldots = 0$

$\tilde{\kappa}_0 z_1 z_5 + \tilde{\kappa}_1 z_1 z_6 + \ldots = 0$

$\tilde{\tilde{\kappa}}_0 z_5^2 + \tilde{\tilde{\kappa}}_1 z_5 z_6 + \ldots = 0$

$$\Downarrow$$

$$z_3^2\, p_{12}(z_{1,2},\, z_{5,\ldots,9}) + z_3 z_4\, q_{12}(z_{1,2},\, z_{5,\ldots,9}) + z_4^2\, r_{12}(z_{1,2},\, z_{5,\ldots,9}) = 0$$

$$\Downarrow$$

$p_{12}(z_{1,2},\, z_{5,\ldots,9}) = q_{12}(z_{1,2},\, z_{5,\ldots,9}) = r_{12}(z_{1,2},\, z_{5,\ldots,9}) = 0$: z_3, z_4 arbitrary $\rightarrow \mathbb{P}^1$

CICY

$$\left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 2 & 0 & 0 \\ \mathbb{P}^4 & 2 & 1 & 2 \end{array}\right]$$

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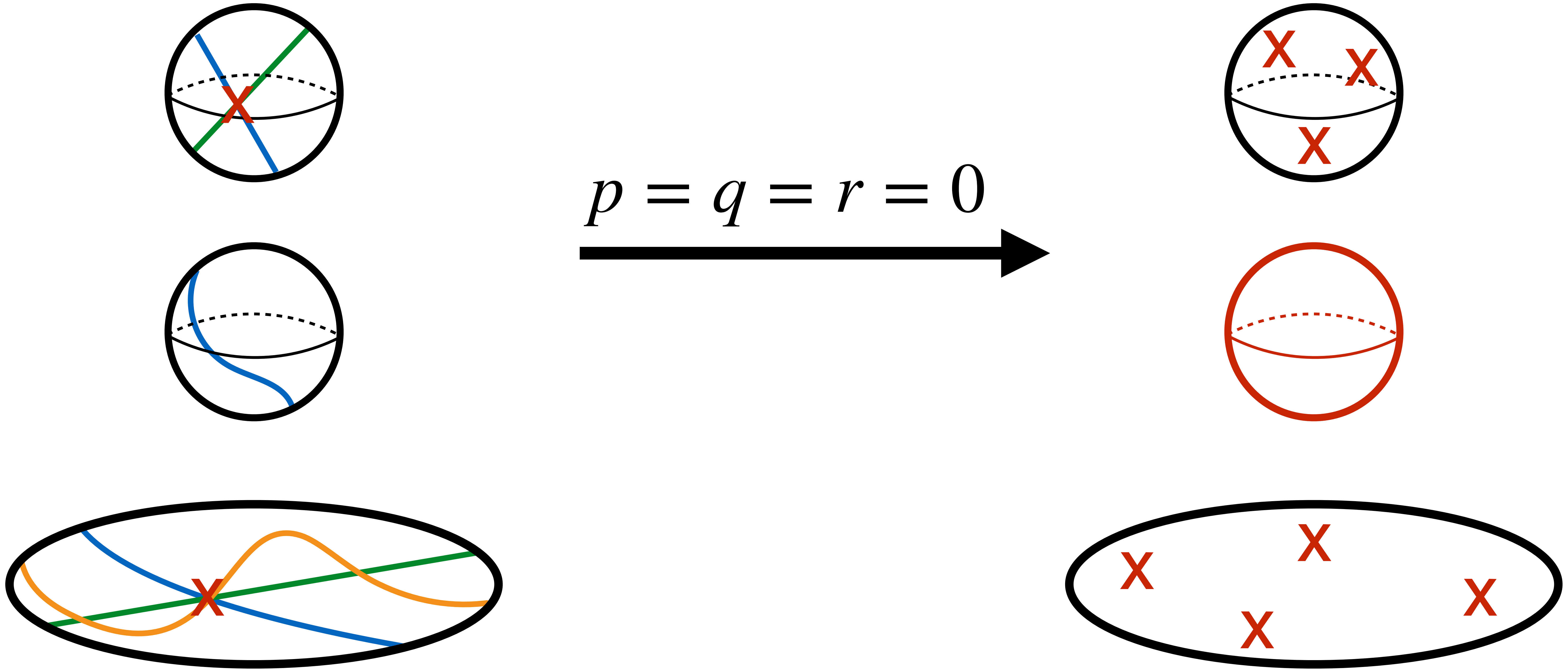

CICY

$$\text{type I}: \; X \in \left[\begin{array}{c|ccccc} \mathbb{P}^1 & 1 & 1 & 0 & \dots & 0 \\ \mathcal{Q} & \hat{q}_1 & \hat{q}_2 & \hat{q}_3 & \dots & \hat{q}_K \end{array}\right], \qquad \text{type II}: \; X \in \left[\begin{array}{c|cccc} \mathbb{P}^1 & 2 & 0 & \dots & 0 \\ \mathcal{Q} & \hat{q}_1 & \hat{q}_2 & \dots & \hat{q}_K \end{array}\right]$$

$$\left[\begin{array}{c|ccccccc} \mathcal{Q} & \hat{q}_1 & \hat{q}_1 & \hat{q}_2 & \hat{q}_2 & \hat{q}_3 & \dots & \hat{q}_K \end{array}\right] \qquad \left[\begin{array}{c|cccccc} \mathcal{Q} & \hat{q}_1 & \hat{q}_1 & \hat{q}_1 & \hat{q}_2 & \dots & \hat{q}_K \end{array}\right]$$

- For all (7'890) CICYs, all single wrapping genus zero GW invariants can be obtained from counting solutions to the equations above.
- Get the position of the curve as a function of CS κ_I

- Why? Maybe because of simplicity of the construction?

Compute the Pfaffian

Monad bundles

- $0 \to \mathcal{O}^{N_f} \xrightarrow{g} A \xrightarrow{f} B \to 0$ w/ $A = \bigoplus_{a=1}^{\mathrm{rk}A} \mathcal{L}(a_1^\alpha, \ldots, a_{h^{11}}^\alpha)\,,\ B = \bigoplus_{\beta=1}^{\mathrm{rk}B} \mathcal{L}(b_1^\beta, \ldots, b_{h^{11}}^\beta)$
- Take $N_f = 0\ :\ 0 \to V \to A \xrightarrow{f} B \to 0$ w/ $f = f(z, \alpha_a)$ a $\mathrm{rk}A \times \mathrm{rk}B$ matrix
- Bundle moduli $\leftrightarrow H^1(X, V \otimes V^*)$. To compute, twist the dual SES with V:
  - $0 \to B^* \otimes V \xrightarrow{f^*} A^* \otimes V \to V^* \otimes V \to 0$
  - Get LES in cohomology (compute ambient space cohomology and restrict to CY via spectral sequence or Koszul resolution)
  - Express in terms of $H^\bullet(B^* \otimes B)\,,\ H^\bullet(C^* \otimes C)\,,\ H^\bullet(C^* \otimes B)$ and extract $f = f(\alpha_a)$
- Pfaff $\leftrightarrow$ $H^0(V_{-1}) = \ker\left[\, f|_{\mathbb{P}^1} : H^0(A \otimes \mathcal{O}(-1)|_{\mathbb{P}^1}) \to H^0(B \otimes \mathcal{O}(-1)|_{\mathbb{P}^1})\,\right]$

Monad bundles - Example

	$\mathcal{Z}_{1,2}$	$\mathcal{Z}_{3,4}$	$\mathcal{Z}_{5,6,7,8,9}$	P_1	P_2	P_3	A_1	A_2	A_3	A_4	B	S	Ξ	
U(1)$_1$	1	0	0	-1	-1	0	1	0	0	0	-1	-1	1	$n_\gamma = 16$
→ U(1)$_2$	0	1	0	-2	0	0	0	1	0	1	-2	0	0	$n_\gamma = 40$
U(1)$_3$	0	0	1	-2	-1	-2	0	0	1	2	-3	-2	2	$n_\gamma = 112$

- $0 \to V \to A \xrightarrow{f} B \to 0$ with $f = (f_{(0,2,3)}\,,\ f_{(1,1,3)}\,,\ f_{(1,2,2)}\,,\ f_{(1,1,1)})$
- Number of bundle moduli: $h^1(X, V \otimes V^*) = 228$
- Restrict and twist by spin bundle: $0 \to V_{-1}|_{\mathbb{P}^1} \to \mathcal{O}(-1) \oplus \mathcal{O}(0) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(0) \to \mathcal{O}(1)$

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- Restrict and twist by spin bundle: $0 \to V_{-1}|_{\mathbb{P}^1} \to \mathcal{O}(-1) \oplus \mathcal{O}(0) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(0) \to \mathcal{O}(1)$
- Parameterize $\mathcal{O}^{\oplus 2} \sim \{\omega_1, \omega_2\}$ and $\mathcal{O}(1) \sim \{z_3, z_4\}$ $\Rightarrow$ write map

$$(\omega_1 \quad \omega_1) \underbrace{\begin{pmatrix} p_{1,3}(z_{1,2}, z_{5,\ldots,9}) & q_{1,1}(z_{1,2}, z_{5,\ldots,9}) \\ r_{1,3}(z_{1,2}, z_{5,\ldots,9}) & s_{1,1}(z_{1,2}, z_{5,\ldots,9}) \end{pmatrix}}_{\delta} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}$$

Monad bundles - Example

$$\delta = \begin{pmatrix} p_{1,3}(z_{1,2}, z_{5,\ldots,9}) & q_{1,1}(z_{1,2}, z_{5,\ldots,9}) \\ r_{1,3}(z_{1,2}, z_{5,\ldots,9}) & s_{1,1}(z_{1,2}, z_{5,\ldots,9}) \end{pmatrix}$$ with

$$p_{1,3}(z_{1,2}, z_{5,\ldots,9}) = \sum_{i=1}^{70} \alpha_i \, m^i_{1,3}(z_{1,2}, z_{5,\ldots,9})$$

$$q_{1,1}(z_{1,2}, z_{5,\ldots,9}) = \sum_{i=1}^{10} \alpha'_i \, m^i_{1,1}(z_{1,2}, z_{5,\ldots,9})$$

$$r_{1,3}(z_{1,2}, z_{5,\ldots,9}) = \sum_{i=1}^{70} \alpha''_i \, m^i_{1,3}(z_{1,2}, z_{5,\ldots,9})$$

$$s_{1,3}(z_{1,2}, z_{5,\ldots,9}) = \sum_{i=1}^{10} \alpha'''_i \, m^i_{1,1}(z_{1,2}, z_{5,\ldots,9})$$

This map degenerates where $\det(\delta) = 0$ → $1'400$ terms bilinear in α

BW: $\sum_{i=1}^{40} \lambda_i \det(\delta(\vec{z}^{\,*}_i)) = 0$ → $1'400 \times 40$ coefficient matrix. This has rank 39!

Extension bundles

- Simplest example — Single extension: $0 \to B \to V \to C \to 0$
- Moduli space: $\mathcal{M}_X(V) = \mathrm{Ext}^1(C, B) \cong H^1(C^* \otimes B)$
- Next step — Double extension: $0 \to A \to V' \to B \to 0\ ,\ \ 0 \to V' \to V \to C \to 0$
- Moduli space: $\mathcal{M}_X(V') = \mathrm{Ext}^1(B, A) \cong H^1(B^* \otimes A)$

$$\mathcal{M}_X(V) = \mathrm{Ext}^1(C, V') \cong H^1(C^* \otimes V')$$

- Compute by twisting first sequence by C^*, look at LES in cohomology, compute ambient space cohomologies, use Koszul resolution to restrict to CY

Extension bundles - Example

- Geometry:

$$X \sim \left[\begin{array}{c|ccc} \mathbb{P}^1 & 1 & 1 & 0 \\ \mathbb{P}^1 & 0 & 0 & 2 \\ \mathbb{P}^2 & 1 & 0 & 2 \\ \mathbb{P}^2 & 0 & 1 & 2 \end{array}\right]^{(4,68)}$$

$$\vec{z}_1 = [z_{1,0} : z_{1,1}]$$
$$\vec{z}_2 = [z_{2,0} : z_{2,1}]$$
$$\vec{z}_3 = [z_{3,0} : z_{3,1} : z_{3,2}]$$
$$\vec{z}_4 = [z_{4,0} : z_{4,1} : z_{4,2}]$$

Bundle:

$$0 \to A \to V \to B \to 0$$
$$A = \mathcal{O}_X(-2,3,-1,1)$$
$$B = \mathcal{O}_X(0,0,2,-2) \oplus \mathcal{O}_X(2,-3,-1,1)$$

- Compute bundle moduli (Note: $h^1(\mathcal{O}_X(a_1 - b_1))$ does not descend from ambient space, but the Pfaffian happens to not depend on these):

$$h^\bullet(\mathcal{O}_\mathcal{P}(a_1 - b_1)) = (0,0,0,40,0,0,0)\,, \qquad h^\bullet(\mathcal{O}_X(a_1 - b_1)) = (0,16,0,0)\,,$$
$$h^\bullet(\mathcal{O}_\mathcal{P}(a_1 - b_2)) = (0,21,0,0,0,0,0)\,, \qquad h^\bullet(\mathcal{O}_X(a_1 - b_2)) = (0,21,25,0)\,.$$

- Focus on first curve class ($n_\gamma = 2$) and look at

$$\underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2}_{\mathcal{A}} \overset{\pi_Q}{\mapsto} \underbrace{\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2}_{Q}$$

Extension bundles - Example

- Using Kunneth, Bott and Serre duality:

$$H^1(\mathcal{O}_{\mathcal{A}}(a_1-b_2)) = H^1(\mathcal{O}_{\mathcal{A}}(4,6,0,0)) = H^1(\mathcal{O}_{\gamma}(-4)) \otimes H^1(\mathcal{O}_{Q}(6,0,0)) \simeq H^0(O_{\gamma}(2))^*$$

- Choose basis $\{r_0^2, r_0 r_1, r_1^2\}$ of $H^0(O_{\gamma}(2))^*$ and expand $v \in H^1(\mathcal{O}_{\mathcal{A}}(-4,6,0,0))$:

$$v = r_0^2 f^{(1)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) + r_0 r_1 f^{(2)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) + r_1^2 f^{(3)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4)$$

- Take extension, tensor with $\mathcal{O}(-1,0,0,0)$, take direct image with π_Q, look at LES, use Bott: $H^0(V_{-1}) \to H^0(\mathcal{O}_{\gamma}(1)\otimes\mathcal{O}_Q(-3,-1,1)) \xrightarrow{f} H^1(\mathcal{O}_{\gamma}(-3)\otimes\mathcal{O}_Q(3,-1,1))$

$$f = \begin{pmatrix} f^{(1)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) & f^{(2)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) \\ f^{(2)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) & f^{(3)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) \end{pmatrix}$$

$$f^{(1)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) = \sum_{i=0}^{6} \beta_i^{(1)} z_{2,0}^i z_{2,1}^{6-i}, \qquad f^{(2)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) = \sum_{i=0}^{6} \beta_i^{(2)} z_{2,0}^i z_{2,1}^{6-i}$$

$$f^{(3)}_{6,0,0}(\vec{z}_2,\vec{z}_3,\vec{z}_4) = \sum_{i=0}^{6} \beta_i^{(3)} z_{2,0}^i z_{2,1}^{6-i}.$$

- Now $\mathrm{Pfaff} \leftrightarrow H^0(V_{-1}) \Rightarrow \det(f) = 0$ is the jumping locus

Comparing GLSM and alg. geom. conditions

Vanishing instanton sum and Hilbert series

Vanishing instanton sum and Hilbert functions

- What causes BW cancelation in algebraic geometry?
  - If I took just 40 arbitrary points, the $1'400 \times 40$ matrix would have rank 40
  - The entries of the coefficient matrix are polynomials in $\vec{z}_i^*$, which themselves are complicated functions of the CS coefficients in the defining equations of the CY
  - The solutions that determine the position of the $\mathbb{P}^1$s as a function of κ are correlated to move on some subvariety that preserves the cancelation
  - Moreover, the number of monomials in these polynomials is linked to the number of vector bundle moduli
- It should be possible to formulate a criterion purely based on (X,V)

Vanishing instanton sum and Hilbert functions

- Think of the complete intersection that specifies the n_γ points where the $\mathbb{P}^1$ lives as a zero-dimensional algebraic variety w/ associated projective ideal $I = \langle P_1, \ldots, P_{K+2} \rangle$
- By going to a patch, we can alternatively think of an associated affine ideal $J = \langle P_1, \ldots, P_{K+2},\ z_{1,0} - 1, \ldots, z_{m,0} - 1 \rangle$
- This induces maps between the associated (projective and affine) coordinate rings: $\mathbb{C}[z] \xrightarrow{r} S \xrightarrow{l} A$ w/ $S = \mathbb{C}[z]/I\,, \quad A = \mathbb{C}[z]/J$
- Note: S and A depend on the polynomials P_i but not on the precise loci of the $\mathbb{P}^1$

Vanishing instanton sum and Hilbert functions

- The coordinate rings $\mathbb{C}[z]$ and $S = \mathbb{C}[z]/I$ are multi-graded with standard Hilbert functions/series: $h_S(k) = \dim(S_k)\,, \quad H_S(t) = \sum_k h_S(k) t^k$ w/ $t^k = t_1^{k_1} \cdots t_m^{k_m}$
- For 0-dimensional varieties: $h_S(k) \to n_\gamma \quad \text{for} \quad k \gg 1$ (compute from syzygies)
- $A = \mathbb{C}[z]/J$ is not graded, but can define a filtration by sub-algebras $A_{\leq k}$ of degree $\leq k$: $h_A(k) = \dim(A_{\leq k}), \quad H_A(t) = \sum_k h_A(k) t^k$ (compute from Grobner)
- Since $A_{\leq k} = l(S_k)$, we have $h_S(k) \geq h_A(k)$, but still $h_S(k) \to n_\gamma \quad \text{for} \quad k \gg 1$

Next, we will see that $h_A(k) = n_\gamma \;\Rightarrow$ Pfaffians lin. indep. $\Rightarrow$ No BW cancelation

Vanishing instanton sum and Hilbert functions

- Consider a polynomial $f \in \mathbb{C}[z]$ with class $[f] = l \circ r(f) \in A$
- Define a linear map $\mu : \mathbb{C}[y] \to \mathrm{End}(A)\,,\quad \mu(f)(a) = [f](a)$
  - $\mu(f)$ acts by mult. in A $\Rightarrow \mu(f) \times \mu(\tilde{f}) = \mu(\tilde{f})\mu(f)$ $\Rightarrow$ lin. maps on A can be simultaneously diagonalized
  - Result in AG: $\{f(z_1^*),\ f(z_2^*),\ \ldots, f(z_{n_\gamma}^*)\} = \{\text{Eigenvalues of } \mu(f)\}$
- We now choose for f a basis of polynomials $(f_I)_{I=1,\ldots,N}$ that appear in the Pfaffian (whose coefficients are the bundle moduli) and consider the $N \times n_\gamma$ coefficient matrix $M_{Ii} = f_I(\vec{z}_i^{\,*})$
  - All these f_I can be diagonalized simultaneously, and the eigenvalues of $\mu(f_I)$ are the entries $(M_{I,1}, \ldots, M_{I,n_\gamma})$
  - Hence $\mathrm{rk}(M) = \dim(\mu(\mathbb{C}[y]_k)) = \dim(l \circ r(\mathbb{C}[y]_k)) = \dim(A_{\leq k}) = h_A(k)$

Comparing GLSM and alg. geom. conditions

Pfaffians and GLSMs

Pfaffians and GLSMs

- Consider (two-term) SU(3) monad bundles
  - Restrict to cases where the monad bundle charges a^i_α and b^i_β are non-negative (better chance for vector bundle to be stable)
  - In this case $h^0(\mathcal{O}_\gamma(-b^i_\beta \cdot w_i - 1)) = 0$ and only spectators can (potentially) decompactify the instanton moduli space
  - Consider cases with a single line bundle for B
  - Focus (wlog) on instanton contributions in cohomology class of first ambient space $\mathbb{P}^1$ factor

$$0 \to V \to \bigoplus_{\alpha=1}^{4} \mathcal{O}_X(a_\alpha, \hat{a}_\alpha) \to \mathcal{O}_X(b, \hat{b}) \to 0$$

Pfaffians and GLSMs

$$0 \to V \to \bigoplus_{\alpha=1}^{4} \mathcal{O}_X(a_\alpha, \hat{a}_\alpha) \to \mathcal{O}_X(b, \hat{b}) \to 0$$

- Part of the (linear) GLSM anomalies impose $c_1(V) = 0$:

$$\sum_{\alpha=1}^{4} a_\alpha = b\,, \qquad \sum_{\alpha=1}^{4} \hat{a}_\alpha = \hat{b}$$

- The (pure quadratic) GLSM anomaly $\mathcal{A}_{1,1} = 0$ imposes

For type I: $\sum_{\alpha<\alpha'} a_\alpha a_{\alpha'} = 0 \quad \Rightarrow a_1 \in \mathbb{Z}_{\geq 0}\,,\ a_2 = a_3 = a_4 = 0$

For type II: $\sum_{\alpha<\alpha'} a_\alpha a_{\alpha'} = 1 \quad \Rightarrow a_1 = a_2 = 1\,,\ a_3 = a_4 = 0$

- The (linear) GLMS anomalies that impose the CY condition $c_1(TX) = 0$ restrict the sum of the charges of the fields defining the geometry to 2

- For type I: spectators will destabilize the vacuum for $a_1 > 2$

  For type II: spectators will not destabilize the vacuum

Pfaffians and GLSMs

- Combined with (mixed quadratic) GLSM anomalies $\mathcal{A}_{1,i} = 0$ we get

For type I: $\quad q_1^i + q_2^i = a_1(b^i - a_1^i)$

For type II: $\quad 2q_1^i = 2b^i - a_1^i - a_2^i$

- More constraints from $\mathcal{A}_{i,j} = 0$ (but I could not use them to infer more model-independent statements)

We observe that the RHS of these anomaly conditions are precisely the degree of the Pfaffian polynomials (continues to hold for more complicated monads, e.g. where $\mathrm{rk}(B){=}2$)

Vanishing of Pfaffians

- From Hilbert series: $h_A(K) = n_\gamma$ for large enough k ⇒ eventually Pfaffians are lin. indep. and cannot cancel a la BW
- On the other hand, we looked at all CICYs with $h^{1,1} = 3$ and scanned over 2-term SU(3) monads w/ $\mathrm{rk}(B)$=1,2. There are over 100 consistent models on more than 30 CICYs
- We find that in ALL cases, even when the spectators have zero modes that can decompactifiy the instanton moduli space, the Pfaffians are lin. dep. Possible explanations:
    1. The examples are so simple that this has to happen.
    2. If we computed the complex coefficients λ_i (e.g. via a residue theorem), we would find their numerical values such that the lin. dep. terms do not cancel
    3. There is some other reason (beyond BW) why the Pfaffians cancel on these spaces
- If 1. is the case, there should be a reason why the terms conspire such that an $N \times n_\gamma$ matrix with $N \gg n_\gamma$ does not have full rank

Conclusions

- From an algebraic point of view, Pfaffians can be computed (up to a complex constant) for all ambient space $\mathbb{P}^1$ curve classes for all CICYs
- For constructions that have a GLSM description, can match onto Bertolini-Plesser
- Can formulate a necessary condition for non-vanishing of the Pfaffian contributions in terms of affine Hilbert functions
- The degree of the Pfaffians appear in the GLSM anomaly conditions
- Checking more than hundred examples, we found that the Pfaffians are linearly dependent, irrespective of whether the instanton moduli space is compact

Thank you very much for your attention!