# "Gauged" Euler Index, Wall-Crossing, and Holonomy Saddles

PILJIN YI
Korea Institute for Advanced Study

GLSM2020@VirginiaTech

Ghim+Hwang+P.Y. 2019

Euler, Morse, and Witten

wall-crossing

holonomy saddles and "gauged" Euler index

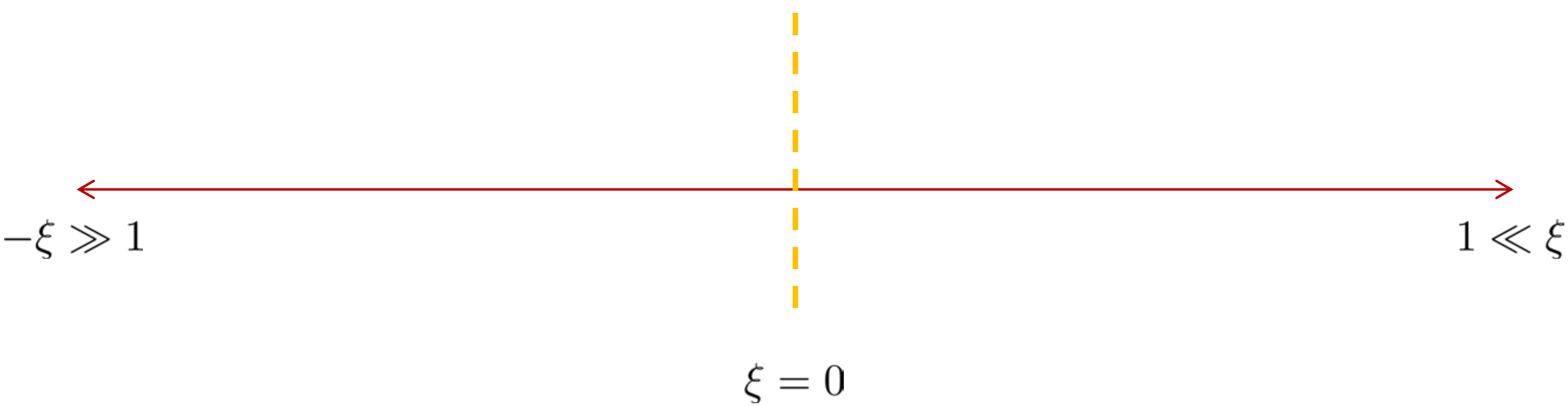
Chern-Simons theories and dualities

# wall-crossing of *d=1* Gauged Linear Sigma Models with *2 (or less)* complex supercharges

$SU(2)_A \times U(1)_V$

gauge fields $(A_0, \Lambda_\pm, X_{1,2,3}, D)^a$ FI constants $\xi$

chirals $(\phi, \Psi_\pm, F)^I$




S.J. Lee + Z.L. Wang + P.Y 2012-13
Hori + H. Kim + P.Y. 2014

# $d$=2 Gauged Linear Sigma Models with (2,2) supersymmetry

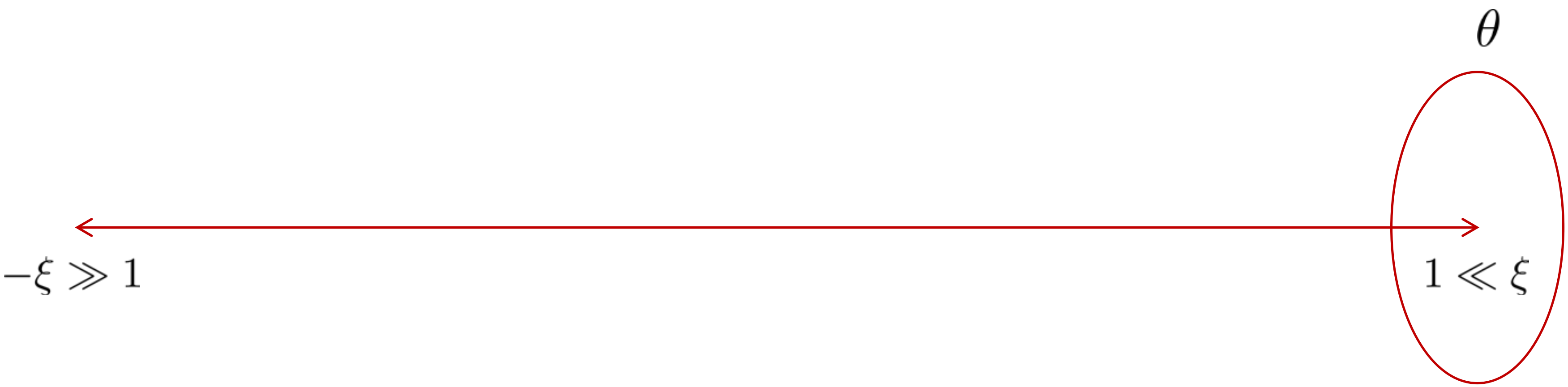
$U(1)_A \times U(1)_V$

$J \qquad R$

gauge fields $(A_{0,1}, \Lambda_{\pm}, \sigma, D)^a$

chirals $(\phi, \Psi_{\pm}, F)^I$

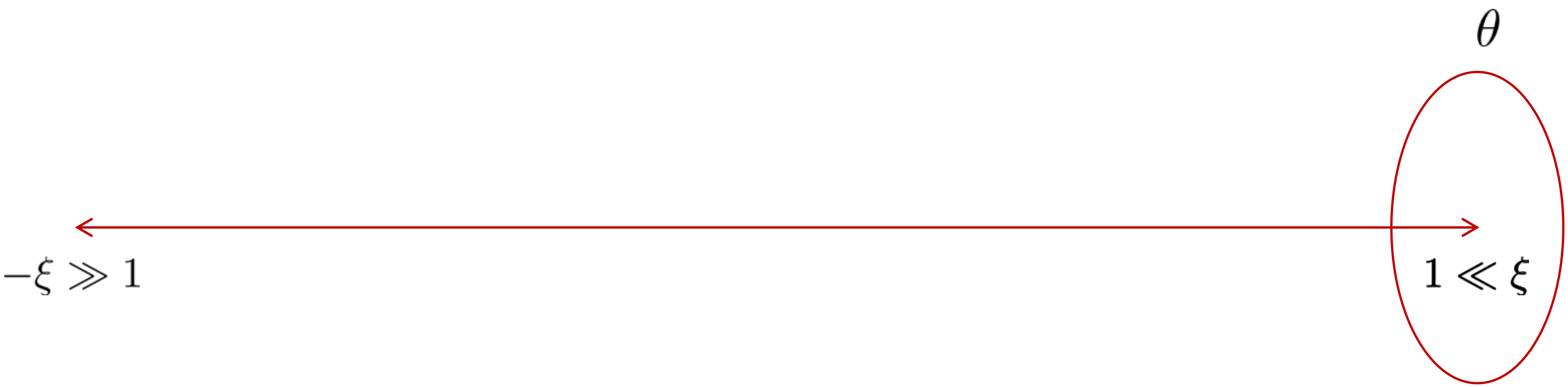
FI constants $\frac{\theta}{2\pi} + i\xi$



# $d$=2 Gauged Linear Sigma Models with (0,2) supersymmetry

$U(1)_R$

| | | |
|---|---|---|
| gauge fields | $(A_{0,1}, \Lambda_-, D)^a$ | FI constants $\dfrac{\theta}{2\pi} + i\xi$ |
| chirals | $(\phi, \Psi_+)^I$ | |
| fermi | $(\Psi_-, F)^I$ | |



twisted partition functions for refined index, *e.g.*,

$$\Omega(\mathbf{y};x) \quad \equiv \quad \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J}\mathbf{y}^{2(R+J)}x^{G_F}e^{-\beta Q^2}\right]$$

*for would-be Hirzebruch genus / Elliptic genus*

$$\mathcal{I}(\mathbf{y};x) \quad \equiv \quad \mathrm{Tr}_{\mathrm{kernel}(Q)}\left[(-1)^{2J}\mathbf{y}^{2(R+J)}x^{G_F}\right]$$

which often translate to Jeffrey-Kirwan contour integrals

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J}\mathbf{y}^{2(J+R)}x^{G_F}e^{-\beta Q^2}\right]$$

$$= \sum \text{JK-Res}_{\eta:\{Q_i\}}\, g(u,\bar{u};0)$$

*generalization of geometric index theorems,*
*via path-integral by Alvarez-Gaume,*
*to gauged systems*

Hori + Kim + P.Y. 2014
Benini + Eager + Hori +Tachikawa 2013
Szenes + Vergne 2004
Brion + M. Vergne 1999
Jeffrey + Kirwan 1993

## *d*=2 GLSM Elliptic Genera

*Benini + Eager + Hori + Tachikawa 2013*

$\xi < 0$ $\xi = 0$ $0 < \xi$

## *d*=1 GLSM Refined Index

*Hori + Kim + P.Y. 2014*

# how about theories with two real supercharges ?

$d=3 \quad \mathcal{N}=1$

$d=2 \quad \mathcal{N}=(1,1)$

$d=1 \quad \mathcal{N}=2a$

one complex supercharge

$d=2 \quad \mathcal{N}=(0,2)$

$d=1 \quad \mathcal{N}=2b$

# basic supermultiplets fall into two types

$$d = 3 \qquad \mathcal{N} = 2$$

$(A_i, \sigma; \Lambda; D)$ $\qquad\qquad$ $(\phi; \Psi; F)$

$\Lambda = (\lambda + i\chi)/\sqrt{2}$

$$d = 3 \qquad \mathcal{N} = 1$$

$(A_i; \lambda)$ $\qquad$ $(\sigma; \chi; D)$ $\qquad$ $\mathrm{Re}(\phi; \Psi; F)$ $\qquad$ $\mathrm{Im}(\phi; \Psi; F)$

vector multiplet

scalar multiplets

auxiliary fields, $D$ and $F$, both belong to scalar multiplets

$$d = 3 \qquad \mathcal{N} = 1$$

$(A_i; \lambda)$

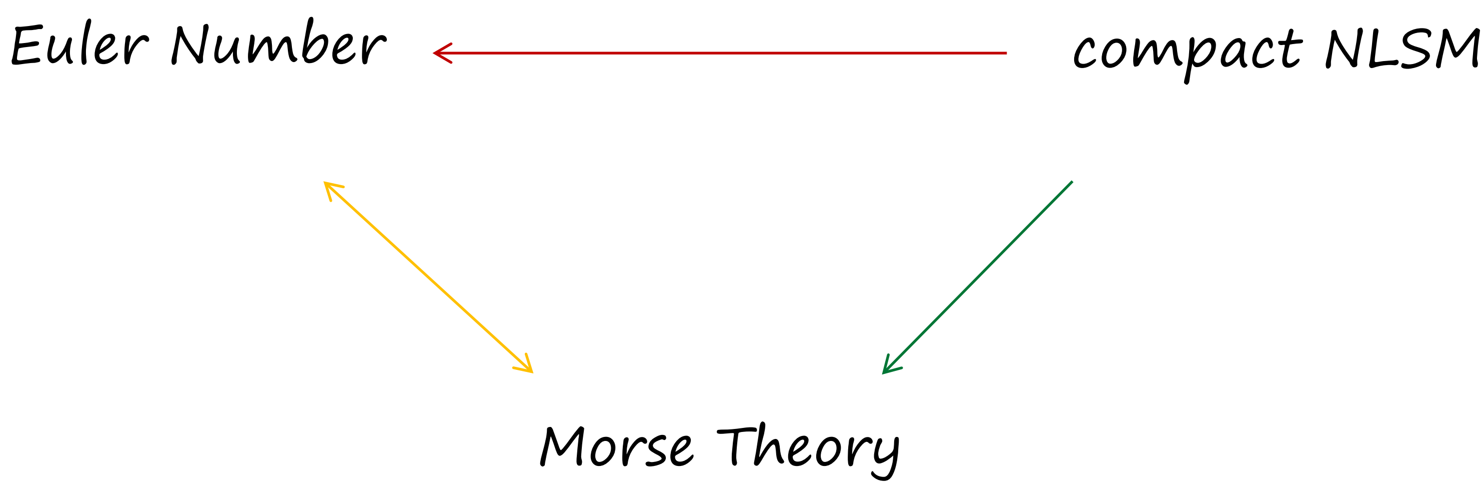
vector multiplets

$(X; \psi; f)$

scalar multiplets

would-be $D$–term wall-crossings now occur along the superpotential side !

we are interested in toroidal twisted partition functions in the small Euclidean time limit, a.k.a., the bulk index, which computes the integral index when the dynamics is fully gapped




Alvarez-Gaume/Witten 1982-1984

# Euler, Morse, and Witten

# nonlinear sigma model

$$\Phi^I = X^I + \theta\psi^I + \theta^2 f^I$$

$$\mathcal{L} = \frac{1}{4}\int d\theta^2 \; g_{JK}(\Phi)\; D^a\Phi^J\, D_a\Phi^K$$

$$D_a = \partial_{\theta_a} + \theta^b \gamma^i_{ab}\partial_i$$

# nonlinear sigma model → differential-form-valued wavefunctions

$$\Phi^I = X^I + \theta\psi^I + \theta^2 f^I$$

$$[\pi_J, X^K] = i\delta_J{}^K + \cdots$$

$$\mathcal{L} = \frac{1}{4}\int d\theta^2\, g_{JK}(\Phi)\, D^a\Phi^J\, D_a\Phi^K$$

$$Q \sim (\psi_1 + i\psi_2)^K \pi_K \qquad\qquad d$$

$$Q^\dagger \sim (\psi_1 - i\psi_2)^K \pi_K \qquad\qquad d^\dagger$$

→ susy algebra → exterior calculus on the target

# nonlinear sigma model → Witten index = Euler number

Alvarez-Gaume 1983

$$\mathcal{I} \;\equiv\; \lim_{\beta\to\infty} \mathrm{Tr}\left((-1)^{\mathcal{F}} e^{-\beta\{Q,Q^\dagger\}}\, x^G\right) \;=\; \pm\sum\,(-1)^p\,\dim H^p$$

$$\mathcal{L} = \frac{1}{4}\int d\theta^2\; g_{JK}(\Phi)\; D^a\Phi^J\, D_a\Phi^K$$

$$Q \sim (\psi_1 + i\psi_2)^K \pi_K \qquad\qquad d$$

$$Q^\dagger \sim (\psi_1 - i\psi_2)^K \pi_K \qquad\qquad d^\dagger$$

susy algebra → exterior calculus on the target

although, in reality, one really computes
the twisted partition function, or the so-called bulk part

$$\Omega \;\equiv\; \mathrm{Tr}\left((-1)^{\mathcal{F}} e^{-\beta\{Q,Q^\dagger\}}\, x^G\right)$$

$$\mathcal{I} \;=\; \Omega\Big|_{\beta\to 0} \;+\; \delta\mathcal{I}_*$$

$$=\; \sum (-1)^p \dim H^p_*$$

for some boundary condition, e.g., $L^2$

with superpotential

$$\Phi^I = X^I + \theta\psi^I + \theta^2 f^I$$

$$\mathcal{L} = \frac{1}{4}\int d\theta^2 \; g_{JK}(\Phi)\; D^a\Phi^J\, D_a\Phi^K \;+\; \int d\theta^2\; W(\Phi)$$

# flat target with superpotential, as a middle step toward GLSM

$$\Phi^I = X^I + \theta\psi^I + \theta^2 f^I$$

$$\mathcal{L} = \frac{1}{4}\int d\theta^2 \; D^a\Phi^K \, D_a\Phi^K \; + \; \int d\theta^2 \; W(\Phi)$$

flat target with superpotential, as a middle step toward GLSM

$$\Omega_W \equiv \mathrm{Tr}\left((-1)^{\mathcal{F}} e^{-\beta\{Q,Q^\dagger\}}\, e^{\mu G}\right)$$

$$\mathcal{L} = \frac{1}{4}\int d\theta^2\; D^a\Phi^K\, D_a\Phi^K \;+\; \int d\theta^2\; W(\Phi)$$

$$\mathcal{H} = \frac{1}{2}\pi^K\pi_K \;+\; \frac{1}{2}\partial_K W\partial^K W \;+\; \psi^J\psi^K\partial_J\partial_K W = \frac{1}{2}\{Q, Q^\dagger\}$$

absence of holomorphy

exact matching of B/F degrees of freedom, unlike (0,2)

no available R-symmetry chemical potential, unlike (2,2)

↓

no determinant factors and no surviving chemical potential

usual “localization” scheme is ineffective the twisted partition function, or the bulk index, is entirely determined by the real superpotential

$$\Omega_W \;\equiv\; \mathrm{Tr}\left((-1)^{\mathcal{F}} e^{-\beta\{Q,Q^\dagger\}}\, e^{\mu G}\right)$$

bosonic zero modes

classical action

$$\lim_{\beta\to 0}\Omega_W = \frac{1}{\sqrt{2\pi}^N}\int d^N x\, \det(\partial_J\partial_K \tilde{W})\, e^{-(\partial\tilde{W})^2/2}$$

from fermionic zero mode integrals

$$X = \beta^{1/2}x + \cdots$$

$$\tilde{W}(x) \equiv W(\Phi \to x)$$

# reduction to an ordinary Gaussian integral
# with all the nontrivial content encoded in the integration ranges

$$\Omega_W \;\equiv\; \mathrm{Tr}\left((-1)^{\mathcal{F}} e^{-\beta\{Q,Q^\dagger\}}\, e^{\mu G}\right)$$

$$\lim_{\beta\to 0}\Omega_W = \frac{1}{\sqrt{2\pi}^N}\int d^N x\, \det(\partial_J\partial_K\tilde{W})\, e^{-(\partial\tilde{W})^2/2}$$

$$= \frac{1}{\sqrt{2\pi}^N}\int d^N Y\; e^{-Y^2/2} \qquad\qquad Y^\mu \equiv \partial_\mu \tilde{W}$$

when the twisted partition function is integral, it is the winding number of the map $x \to \partial\tilde{W}$ btw two Euclidean spaces

$$\partial W \ : \ \mathbf{R}^n \ \to \ \tilde{\mathbf{R}}^n$$

$$\lim_{\beta\to 0} \Omega_W = \frac{1}{\sqrt{2\pi}^N} \int d^N x \, \det(\partial_J \partial_K \tilde{W}) \, e^{-(\partial\tilde{W})^2/2}$$

$$= \frac{1}{\sqrt{2\pi}^N} \int d^N Y \ e^{-Y^2/2}$$

if integral, the Morse theory

Witten 1982

# the superpotential is the Morse function

$$\partial W \; : \; \mathbf{R}^n \; \rightarrow \; \tilde{\mathbf{R}}^n$$

$$\mathcal{I}_W = \lim_{\beta \to 0} \Omega_W = \frac{1}{(2\pi)^{n/2}} \int_{\partial W(\mathbf{R}^n)} d^n Y \; e^{-Y^2/2}$$

$$= \lim_{C \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{C \cdot \partial W(\mathbf{R}^n)} d^n(CY) \; e^{-C^2 Y^2/2}$$

$$= \sum_{\partial W(x_*)=0} \mathrm{sgn}(\det(\partial\partial W(x_*)))$$

wall-crossing, or not

with $N = 1$, for a polynomial $W$

$$W = a_0 \Phi^{k+1} + a_1 \Phi^k + \cdots + a_{k+1}$$

$$\lim_{\beta \to 0} \Omega_W = \frac{1}{\sqrt{2\pi}} \int dx \, \det(\partial_\mu \partial_\nu \tilde{W}) e^{-(\partial \tilde{W})^2/2}$$

$$= \begin{cases} \frac{1}{\sqrt{2\pi}} \int_{\pm R^1} dY \, e^{-Y^2/2} & \text{odd } k \\ 0 & \text{even } k \end{cases}$$

$$= \begin{cases} \mathrm{sgn}(a_0) & \text{odd } k \\ 0 & \text{even } k \end{cases}$$

wall-crossing

$Y \equiv \tilde{W}'(x)$

with $N=2$ and a superpotential which is the real part of a holomorphic polynomial of degree $k+1$, the map $x \to \partial\tilde{W}$ is $-k$ -fold cover of $C^1$,

$$z = x_1 + i x_2$$

$$\tilde{W}(x_I) = \mathrm{Re}(b_0 z^{k+1} + \cdots + b_{k+1})$$

$$\mathcal{I}_W = \lim_{\beta\to\infty} \Omega_W = \lim_{\beta\to 0} \Omega_W = \frac{1}{2\pi}\int dz\, d\bar{z}\, \det(\partial_\mu \partial_\nu \tilde{W})\, e^{-(\partial\tilde{W})^2/2}$$

$$= \frac{1}{2\pi}\int_{-C^1\cup\cdots\cup -C^1} dZ\, d\bar{Z}\; e^{-Z\bar{Z}/2}$$

$$= -k$$ no wall-crossing

$$Z \equiv \partial_z \tilde{W}$$

with $N = 2$ and a generic real polynomial $W$ of degree $k+1$ ,

$$\mathcal{I}_W = \lim_{\beta \to 0} \Omega_W = \; -k,\; -k+2,\; -k+4,\; \ldots\;,\; -k+2[(k+1)/2]$$

wall-crossing

with odd $N$ and a generic real polynomial $W$ of odd degree $k+1$,

$$\mathcal{I}_W = \lim_{\beta\to\infty} \Omega_W = \lim_{\beta\to 0} \Omega_W = 0$$

no wall-crossing

for multiple free massive scalars

$$W = \frac{1}{2} m_{JK} \Phi^J \Phi^K$$

$$\mathcal{I}_W = \lim_{\beta \to \infty} \Omega_W = \lim_{\beta \to 0} \Omega_W = \frac{1}{\sqrt{2\pi}^N} \int_{-\infty}^{\infty} d^N x \, \det(m) \, e^{-(m^t m)_{\mu\nu} x^\mu x^\nu / 2}$$

$$= \frac{1}{\sqrt{2\pi}^N} \int d^N Y \, \mathrm{sgn}(\det(m)) e^{-Y^2/2}$$

$$= \mathrm{sgn}(\det(m))$$

wall-crossing

why wall-crossing ?

# the simplest wall-crossing

$$W = \frac{1}{2} m \Phi^2$$

$$\mathcal{I}_W = \lim_{\beta \to \infty} \Omega_W = \lim_{\beta \to 0} \Omega_W = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \; m \, e^{-(mx)^2/2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\mathrm{sgn}(m)\infty}^{\mathrm{sgn}(m)\infty} dY \; e^{-Y^2/2}$$

$$= \mathrm{sgn}(m)$$

wall-crossing

free massive case can be smoothened in the toy model of the form

$$W = \frac{m\Phi^2}{1+\sqrt{\lambda^2\Phi^2}}$$

$$\lim_{\beta\to 0}\Omega_W = \frac{1}{\sqrt{2\pi}}\int_{-m/|\lambda|}^{m/|\lambda|} dY\, e^{-Y^2/2} \quad = \quad \mathrm{Erf}(m/|\lambda|)$$


$$x \to \partial\tilde{W}$$

what do we do if the covering is not complete?

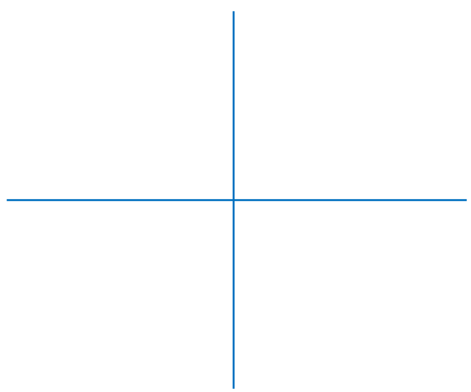
$$\partial W \ : \ \mathbf{R}^n \hookrightarrow \tilde{\mathbf{R}}^n$$

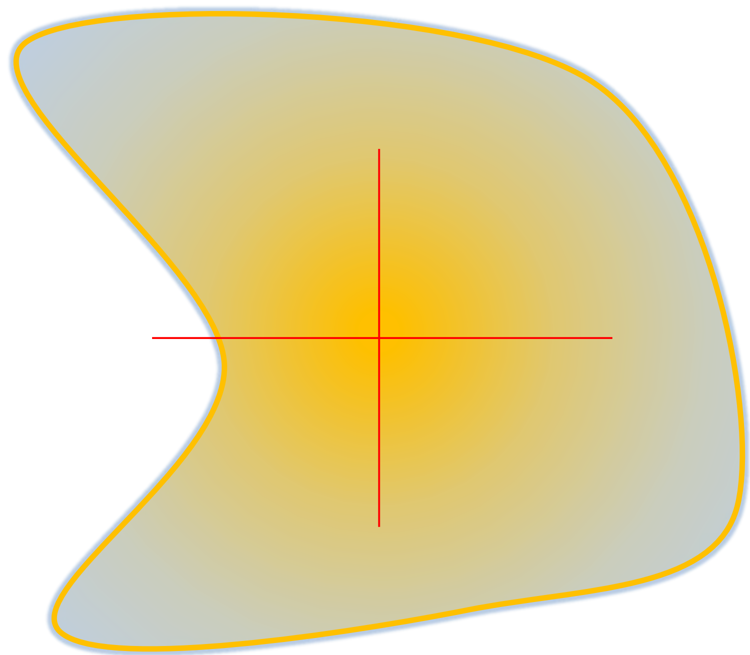
includes the origin $N_{\mathrm{c}}$-times
with orientation taken into account

integral witten index must be computed with a scaled up superpotential if the covering is not complete

$$\partial(C \cdot W) \ : \ \mathbf{R}^n \hookrightarrow \tilde{\mathbf{R}}^n$$

includes the origin multiple times
with orientations taken into account





# integral witten index must be computed with a scaled up superpotential if the covering is not complete

$$\partial(C \cdot W) \ : \ \mathbf{R}^n \hookrightarrow \tilde{\mathbf{R}}^n$$

includes the origin $\mathcal{I}_W$-times
with orientation taken into account

$$\lim_{\beta \to 0} \Omega_{C \cdot W} \ \equiv \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} d^n x \ \det(\partial\partial(C \cdot W)) e^{-(\partial C \cdot W)^2/2}$$

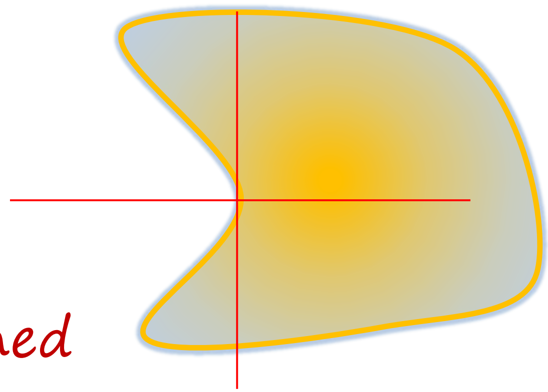
$$\mathcal{I}_W = \lim_{C \to \infty} \lim_{\beta \to 0} \Omega_{C \cdot W} \ = \sum_{\partial W(x_*)=0} \mathrm{sgn}(\det(\partial\partial W(x_*)))$$

what happens if there is an asymptotic flat direction?
the above scaling cannot lift the asymptotics, nor is it desirable

$$\partial W \ : \ \mathbf{R}^n \hookrightarrow \tilde{\mathbf{R}}^n$$

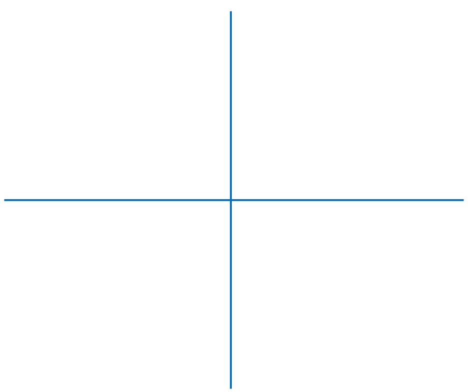
$$\lim_{C\to\infty} \lim_{\beta\to 0} \Omega_{C\cdot W}$$ *non-integral ?*

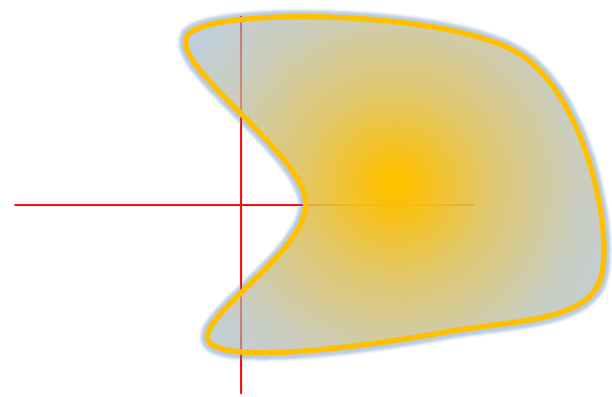
$$\sum_{\partial W(x_*)=0} \mathrm{sgn}(\det(\partial\partial W(x_*)))$$ *ill-defined*

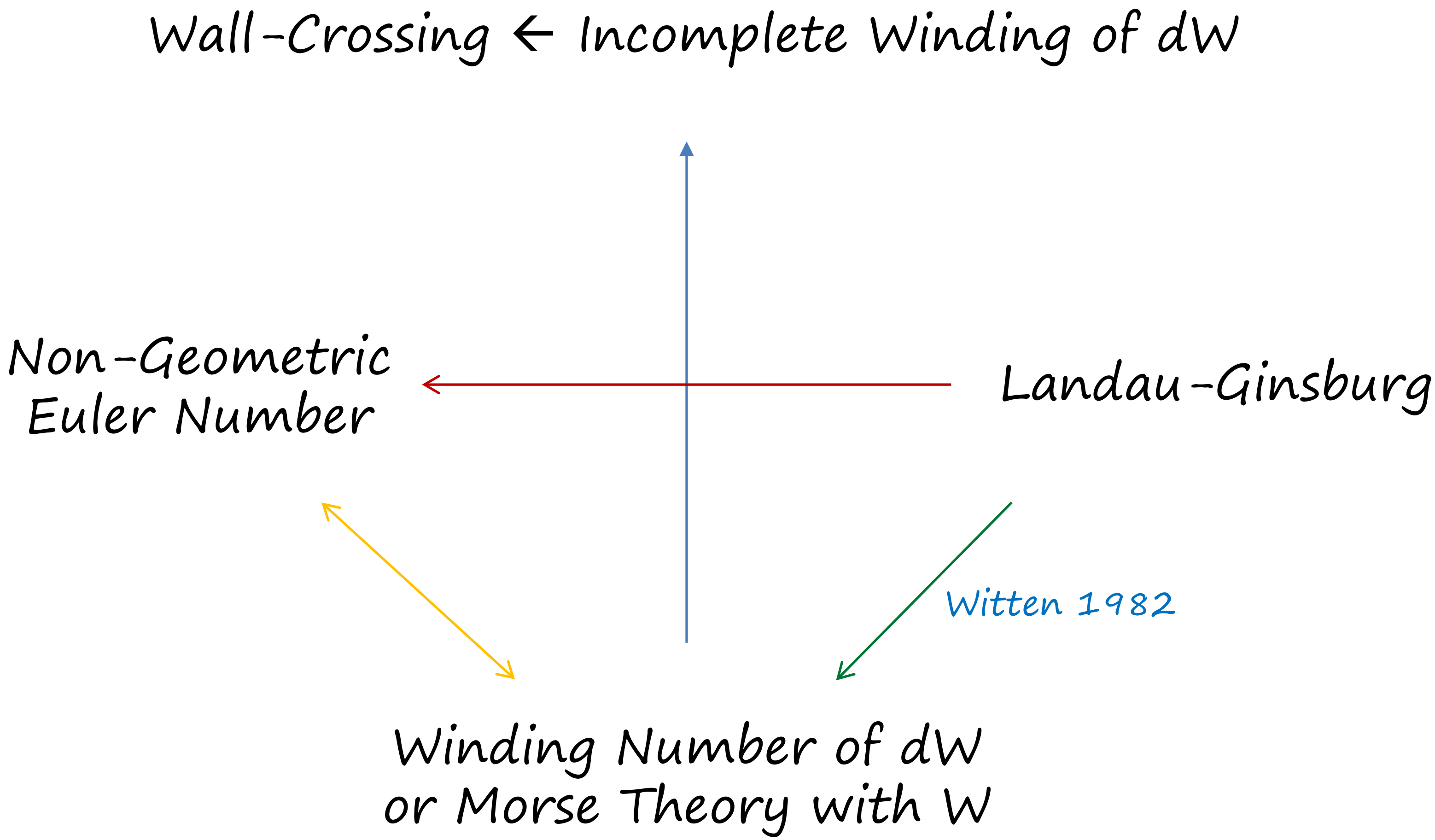


the origin moves inside/outside → wall-crossing!

$$\partial W \ : \ \mathbf{R}^n \hookrightarrow \tilde{\mathbf{R}}^n$$



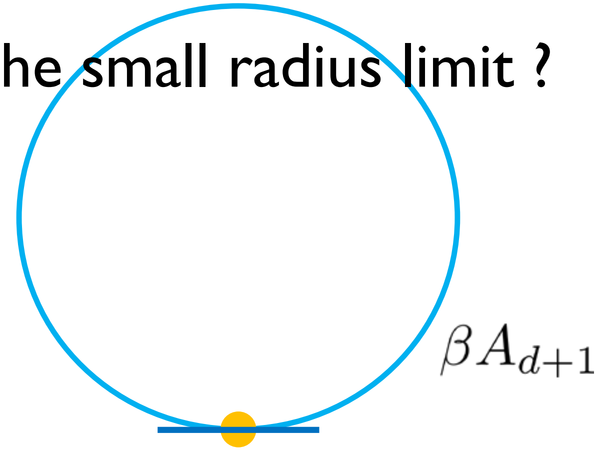



Wall-Crossing ← Incomplete Winding of dW
Non-Geometric
Euler Number
Landau-Ginsburg
Witten 1982
Winding Number of dW
or Morse Theory with W

everything up to this point is applicable straightforwardly to all dimensions up to d=3, since the computation proceeds via toroidal compactification; also, these could have been inferred, with some care, from classic literatures from 1980's

but things change qualitatively if a gauge sector is introduced holonomy saddles and gauged Euler index

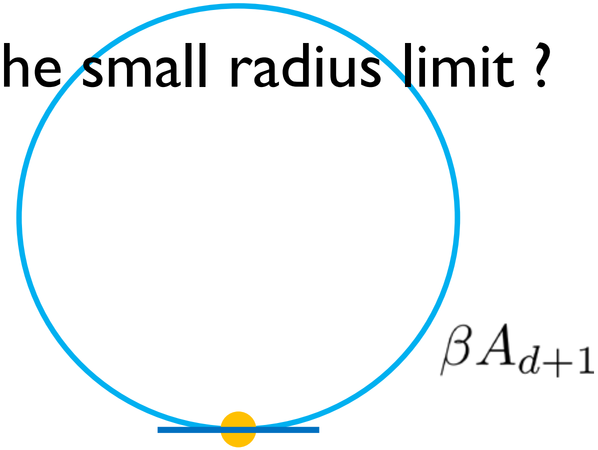
## index, or twisted partition functions in the small radius limit ?




$$\Omega^{G}_{d+1}\Big|_{\beta\to 0} \overset{??}{=} \Omega^{G}_{d}$$

twisted partition function in $d=d+1$
vs
(twisted) partition function in $d=d$ ??

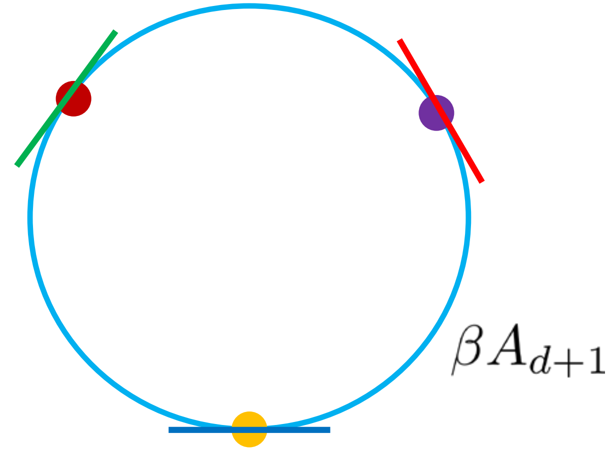
## index, or twisted partition functions in the small radius limit ?




$$\Omega^G_{d+1}\Big|_{\beta\to 0} \neq \Omega^G_d$$

not a big surprise, since Witten index is known
to be not preserved upon dimensional reductions

exactly what is the underlying mechanism, against the
naïve topological invariance of twisted partition functions ?




$$\Omega^{G}_{d+1}\Big|_{\beta\to 0} = \sum_{H} c_{G:H}\,\Omega^{H}_{d}$$

at some special holonomy values,
separated from origin at a finite distance,
a theory with smaller field content, with its
own twisted partition function, may reside
and contribute additively to the left hand side

→ holonomy saddles

Hwang, P.Y. 2017

note that localization of twisted partition functions

all secretly takes small radius limit

$$\beta \to 0$$

the holonomy saddles are relevant

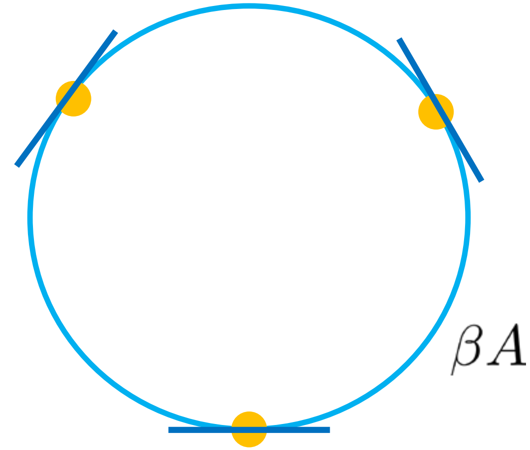
for all types of twisted partition functions

prototype example, with four supercharges:
SU(N) Yang-Mill quantum mechanics vs SU(N) matrix model

$$\Omega_{d=1}^{SU(N)}\Big|_{\beta\to 0} = N \times \mathcal{Z}^{SU(N)} = \mathcal{Z}^{SU(N)/Z_N}$$

Hwang, P.Y. 2017

# SU(N)



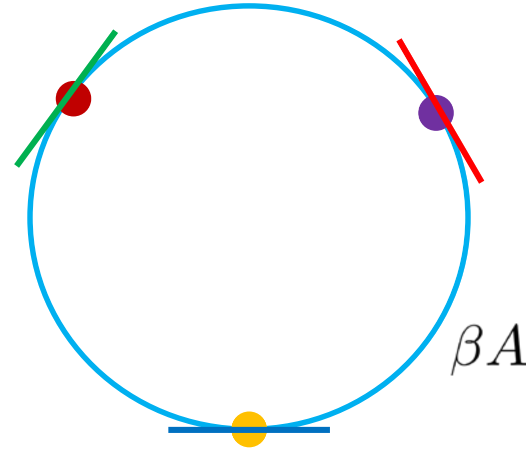

$$\Omega^{G}_{d=1}\Big|_{\beta\to 0} = \sum_{\Theta^*_{SU(N)}} \int dZ\, d\Phi\; e^{-[Z,Z]^2/4 + Z_\mu K_\mu(\Phi)/2}$$

$$= N \times \int dZ\, d\Phi\; e^{-[Z,Z]^2/4 + Z_\mu K_\mu(\Phi)/2}$$

$$= \mathcal{Z}^{SU(N)/Z_N} \quad \equiv \mathcal{Z}^{su(N)}$$

Hwang, P.Y. 2017

more generally




$$\Omega^G_{d=1}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^G(z') + \sum_{H<G} c_{G:H}\,\frac{|\det(Q^G)|/|W_G|}{|\det(Q^H)|/|W_H|}\,\mathcal{Z}^H(z')$$

holonomy saddles ← no decoupled U(1) factor in the low-energy effective theory

Hwang, P.Y. 2017

Hwang, P.Y. 2017

$$\Omega^{SU(N)}_{d=1}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{su(N)}(z')$$

$$\Omega^{Sp(K)}_{d=1}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{sp(K)}(z') + \sum_{m=1}^{K-1}\frac{1}{4}\,\mathcal{Z}^{sp(m)\times sp(K-m)}(z')$$

$$\Omega^{SO(2N)}_{d=1}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{so(2N)}(z') + \sum_{m=2}^{N-2}\frac{1}{8}\,\mathcal{Z}^{so(2m)\times so(2N-2m)}(z')$$

$$\Omega^{SO(2N+1)}_{d=1}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{so(2N+1)}(z') + \sum_{m=2}^{N}\frac{1}{4}\,\mathcal{Z}^{so(2m)\times so(2N+1-2m)}(z')$$

$$\Omega^{G}_{d=1}(\mathbf{y}) = \Omega^{G/\mathcal{C}_G}_{d=1}(\mathbf{y})$$

$$\mathcal{Z}^{h} \equiv \mathcal{Z}^{H/\mathcal{C}_H}$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

P.Y. /
Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000
Pestun 2002

| $\mathcal{N}=4$ | $\mathcal{I}^G_{\text{bulk}}=\Omega^G$ | $\mathcal{I}^G_{\text{bulk}}=-\delta\mathcal{I}^G$ | ~~$\mathcal{I}^G_{\text{bulk}}$~~ $=\mathcal{Z}^G$ |
|---|---|---|---|
| $SU(N)$ | $\frac{1}{N^2}$ | $\frac{1}{N^2}$ | $\frac{1}{N^2}$ |
| $Sp(2)$ | $\frac{5}{32}$ | $\frac{5}{32}$ | $\frac{9}{64}$ |
| $Sp(3)$ | $\frac{15}{128}$ | $\frac{15}{128}$ | $\frac{51}{512}$ |
| $Sp(4)$ | $\frac{195}{2048}$ | $\frac{195}{2048}$ | $\frac{1275}{16384}$ |
| $Sp(5)$ | $\frac{663}{8192}$ | $\frac{663}{8192}$ | $\frac{8415}{131072}$ |
| $Sp(6)$ | $\frac{4641}{65536}$ | $\frac{4641}{65536}$ | $\frac{115005}{2097152}$ |
| $Sp(7)$ | $\frac{16575}{262144}$ | $\frac{16575}{262144}$ | $\frac{805035}{16777216}$ |
| $SO(7)$ | $\frac{15}{128}$ | $\frac{15}{128}$ | $\frac{25}{256}$ |
| $SO(8)$ | $\frac{59}{1024}$ | $\frac{59}{1024}$ | $\frac{117}{2048}$ |
| $SO(9)$ | $\frac{195}{2048}$ | $\frac{195}{2048}$ | $\frac{613}{8192}$ |
| $SO(10)$ | $\frac{27}{512}$ | $\frac{27}{512}$ | $\frac{53}{1024}$ |
| $SO(11)$ | $\frac{663}{8192}$ | $\frac{663}{8192}$ | $\frac{1989}{32768}$ |
| $SO(12)$ | $\frac{1589}{32768}$ | $\frac{1589}{32768}$ | $\frac{6175}{131072}$ |
| $SO(13)$ | $\frac{4641}{65536}$ | $\frac{4641}{65536}$ | $\frac{26791}{524288}$ |
| $SO(14)$ | $\frac{1471}{32768}$ | $\frac{1471}{32768}$ | $\frac{5661}{131072}$ |
| $SO(15)$ | $\frac{16575}{262144}$ | $\frac{16575}{262144}$ | $\frac{92599}{2097152}$ |
| $G_2$ | $\frac{35}{144}$ | $\frac{35}{144}$ | $\frac{151}{864}$ |
| $F_4$ | $\frac{30145}{165888}$ | $\frac{30145}{165888}$ | $\frac{493013}{3981312}$ |

$$\mathcal{I}^G_{\text{bulk}} = \Omega^G\Big|_{\beta\to 0} \quad\longrightarrow\quad = \sum \# \cdot \mathcal{Z}^h$$

$$= -\delta\mathcal{I}^G$$

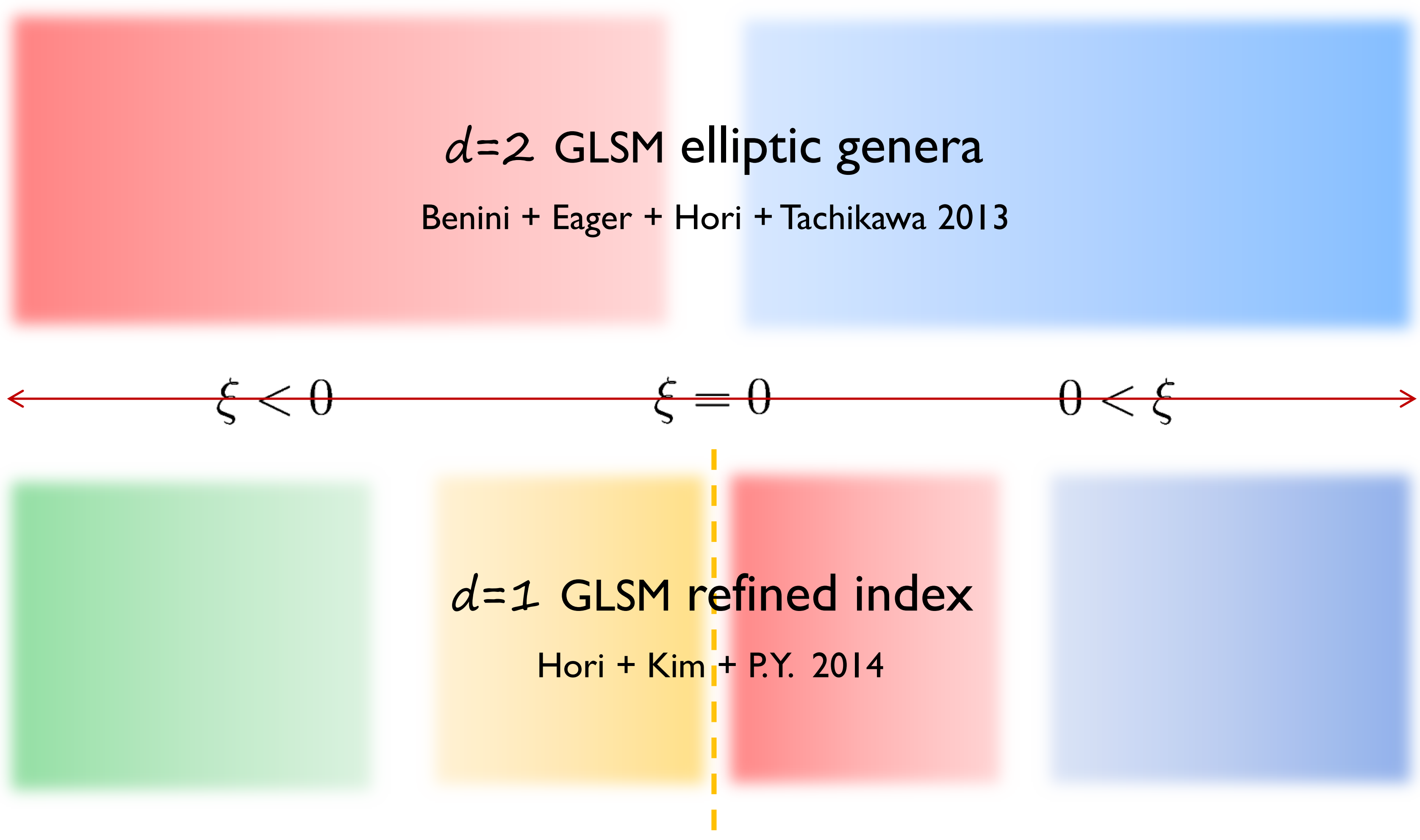
S.J. Lee, P.Y. 2015/2016

Hwang, P.Y. 2017

holonomy saddles appear because
the periodic gauge holonomy variables must be integrated over
and thus are relevant for all space-time dimensions;

they explain many of subtleties in relating partition functions
of susy gauge theories in two adjacent dimensions,
such as how dualities are not straightforwardly
preserved under dimensional reduction

# holonomy saddles also explains why $d$=1 GLSM wall-crosses while $d$=2 GLSM does not



holonomy saddles appears because
the periodic gauge holonomy variables must be integrated over
and thus are relevant regardless of space-time dimensions;

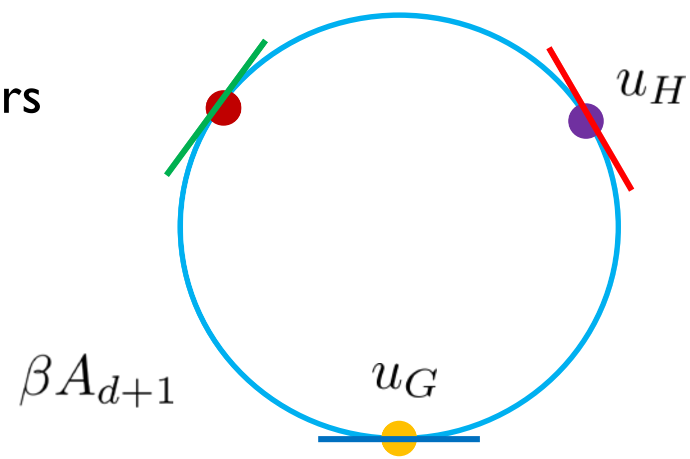
holonomy saddles also allow us to relate
Witten indices of gauge theories across dimensions

# the bulk part of Witten indices can be related across dimensions systematically

$S^1 \times T^d$ $\qquad\qquad\qquad\qquad$ $T^d$

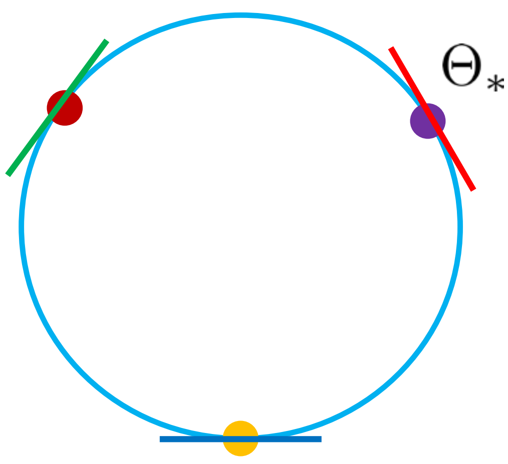
$$\Omega^G_{d+1}(\beta\tilde{z})\Big|_{\beta\to 0} \;\to\; \sum_{u_H} c_{G:H}\,\Omega^H_d(\tilde{z})$$

purely algebraic factors



in particular, for $d=1$ susy gauge theories

$$\lim_{\beta \to 0} \Omega = \sum_{\Theta_*}^{1d} \mathcal{Z}^{\Theta_*}$$

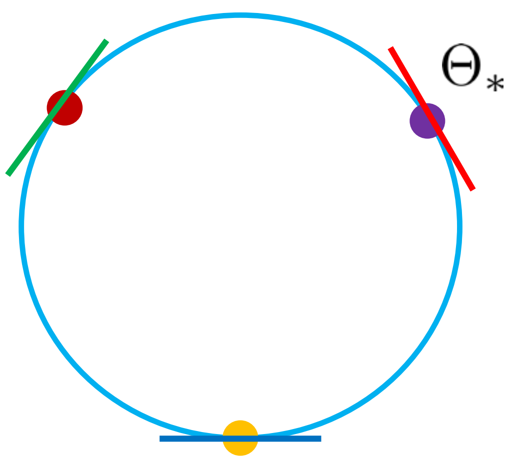


in particular, for $d$=1 susy gauge theories

$$\lim_{\beta\to 0}\Omega = \sum_{\Theta_*}^{1d} \mathcal{Z}^{\Theta_*} = \sum_{\Theta_*}^{1d}\sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*}$$

holonomy saddles

light charged scalar multiplets
at the given holonomy saddle



## with a single gauge multiplet and multiple scalars

$$\lim_{\beta\to 0}\Omega = \sum_{\Theta_*}^{1d} \mathcal{Z}^{\Theta_*} = \sum_{\Theta_*}^{1d} \sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*}$$

holonomy saddles

light charged scalar multiplets
at the given holonomy saddle

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{|q_I|\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3v \int d^{|\mathcal{S}|}s \int d|z_I| \prod_{K\in\mathcal{N}_*}^{K\neq I} d(y_K^2/2)\ \mathrm{sgn}(W_I)\, e^{-V}$$

$$V \equiv \frac{1}{2}\left(\vec{v}^2 + \sum_{\mathcal{S}} s^2 + |z_I| + \sum_{K\in A_*}^{K\neq I} y_K^2\right)$$

all contributions reduce to ordinary Gaussian integrals, and the relevant details are all stored into the integration domain

$$\vec{v} \equiv |q_I||x_I|\vec{u}$$

$$W = W(P; X_J^2)$$

$$s \equiv \partial_P W$$

$$W_J \equiv \partial_{X_J^2/2} W$$

$$z_I \equiv X^I W_I$$

$$y_K \equiv X^K \sqrt{q_K \vec{u}^2 + (W_K)^2}$$

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{|q_I|\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3v \int d^{|\mathcal{S}|}s \int d|z_I| \prod_{K\in\mathcal{N}_*}^{K\neq I} d(y_K^2/2)\ \mathrm{sgn}(W_I)\, e^{-V}$$

$$V \equiv \frac{1}{2}\left(\vec{v}^2 + \sum_{\mathcal{S}} s^2 + |z_I| + \sum_{K\in A_*}^{K\neq I} y_K^2\right)$$

# prototype : multiple massive charged scalars

$$\vec{v} \equiv |q_I||x_I|\vec{u}$$

$$W = W(P; X_J^2)$$

$$= \frac{1}{2}\sum_J m_J X_J^2$$

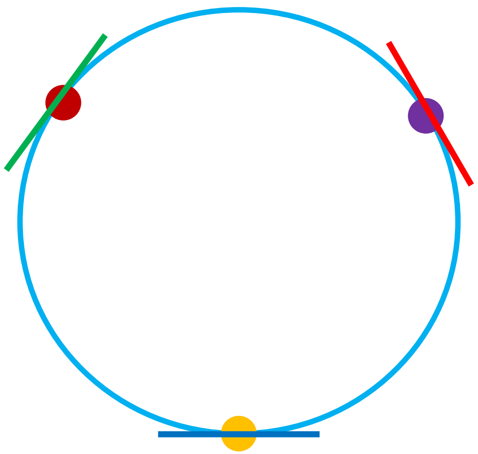
$$z_I \equiv X^I W_I$$

$$y_K \equiv X^K \sqrt{q_K \vec{u}^2 + (W_K)^2}$$

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{|q_I|\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3v \int d^{|\mathcal{S}|}s \int d|z_I| \prod_{K\in\mathcal{N}_*}^{K\neq I} d(y_K^2/2)\,\mathrm{sgn}(W_I)\,e^{-V}$$

$$= \frac{\mathrm{sgn}(m_I)}{2|q_I|}$$

## which shows how holonomy saddles work



$$\lim_{\beta\to 0}\Omega = \sum_{\Theta_*}^{1d} \mathcal{Z}^{\Theta_*} = \sum_{\Theta_*}^{1d}\sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*}$$

$$= \sum_J |q_J| \times \frac{\mathrm{sgn}(m_J)}{2|q_J|} = \sum_J \frac{\mathrm{sgn}(m_J)}{2}$$

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{|q_I|\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3v \int d^{|\mathcal{S}|}s \int d|z_I| \prod_{K\in\mathcal{N}_*}^{K\neq I} d(y_K^2/2)\ \mathrm{sgn}(W_I)\, e^{-V}$$

$$= \frac{\mathrm{sgn}(m_I)}{2|q_I|}$$

Coulombic continuum

prototype : single neutral + multiple charged scalars

$$W = P \cdot \left( \sum_J q_J X_J^2 - \xi \right)$$

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{|q_I|\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3 v \int d^{|\mathcal{S}|} s \int d|z_I| \prod_{K \in \mathcal{N}_*}^{K \neq I} d(y_K^2/2)\, \mathrm{sgn}(W_I)\, e^{-V}$$

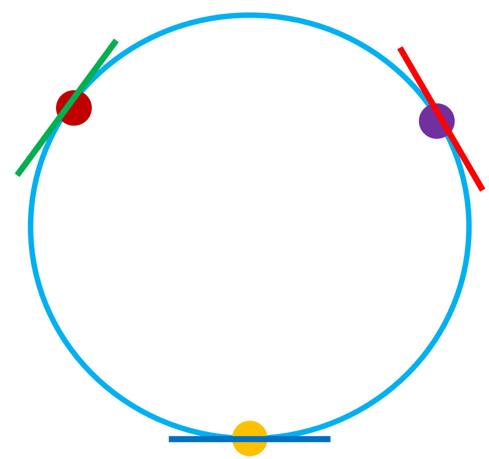
$$= \left( \begin{array}{cc} -1/|q_I| & q_I \xi > 0 \\ & \\ 0 & q_I \xi_I < 0 \end{array} \right.$$

# WCP ← single neutral + multiple positively charged scalars

$$\lim_{\beta\to 0}\Omega = \sum_{\Theta_*}^{1d} \mathcal{Z}^{\Theta_*} = \sum_{\Theta_*}^{1d} \sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*} = \sum_I |q_I| \mathcal{Z}_I \quad = \begin{pmatrix} -N_f & \xi > 0 \\ 0 & \xi_I < 0 \end{pmatrix}$$

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{q_I\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3 v \int d^{|\mathcal{S}|} s \int d|z_I| \prod_{K\in\mathcal{N}_*}^{K\neq I} d(y_K^2/2)\,\mathrm{sgn}(W_I)\, e^{-V}$$

$$= \begin{pmatrix} -1/q_I & \xi > 0 \\ 0 & \xi_I < 0 \end{pmatrix}$$



what happens if you elevate the WCP prototype to $d=3$ ?

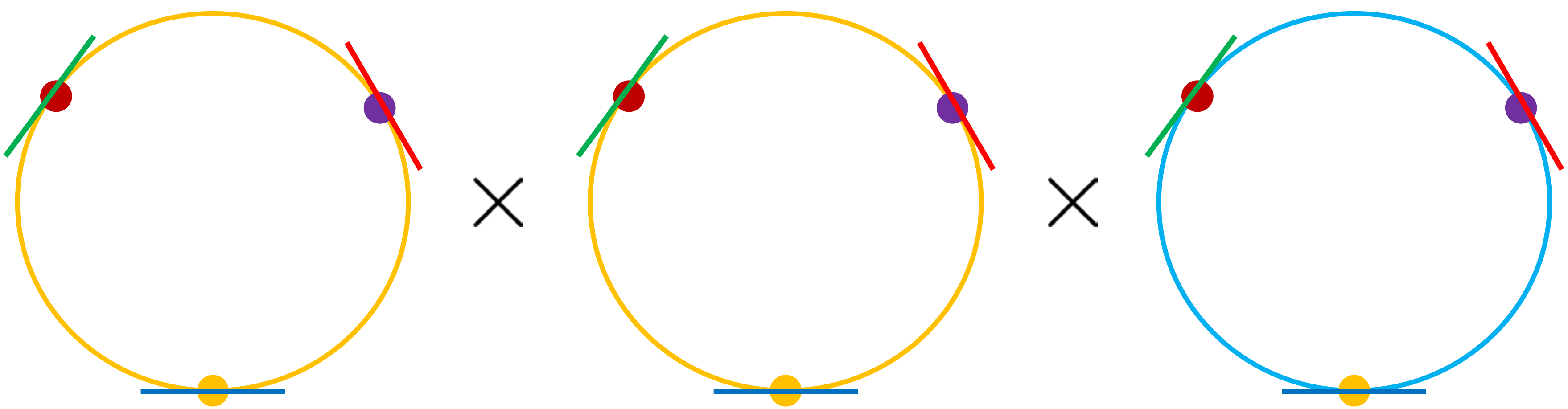
$$W = P \cdot \left( \sum_J q_J X_J^2 - \xi \right)$$

$$\mathcal{Z}_I^{\Theta_*} = \frac{1}{|q_I|\,(2\pi)^{2+|\mathcal{S}|/2}} \int d^3 v \int d^{|\mathcal{S}|} s \int d|z_I| \prod_{K \in \mathcal{N}_*}^{K \neq I} d(y_K^2/2)\, \mathrm{sgn}(W_I)\, e^{-V}$$

$$= \left( \begin{array}{cc} -1/|q_I| & q_I \xi > 0 \\ 0 & q_I \xi_I < 0 \end{array} \right.$$

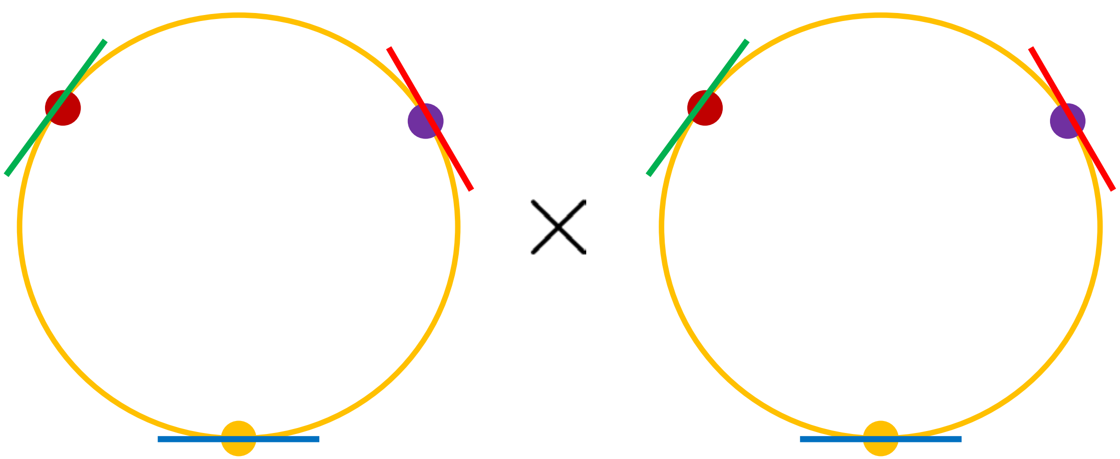
holonomy saddles proliferate with increasing dimensions !

$$\lim_{\beta\to 0}\Omega^{3d} = \sum_{\Theta_*}^{3d} \mathcal{Z}^{\Theta_*} = \sum_{\Theta_*}^{3d}\sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*} = \sum_J |q_J|^3 \mathcal{Z}_J = \left( \begin{array}{ll} -\sum_J q_J^2 & \xi > 0 \\ 0 & \xi < 0 \end{array} \right.$$



the resulting vacuum degeneracy can be also understood as being due to discrete holonomies along spatial circles

$$\lim_{\beta\to 0}\Omega^{3d} = \sum_{\Theta_*}^{3d} \mathcal{Z}^{\Theta_*} = \sum_{\Theta_*}^{3d}\sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*} = \sum_J |q_J|^3 \mathcal{Z}_J = \left(\begin{array}{ll} -\sum_J q_J^2 & \xi > 0 \\ 0 & \xi < 0 \end{array}\right.$$



discrete $Z_{q_J}$ holonomies along the two spatial circles, from the Hamiltonian side

for d=2 and for d=3 without Chern-Simons,
the index computation is uplifted straightwardly

the only difference is that the holonomy saddles
proliferate further, since there are more circles,
hence more of dual holonomy circles

for $d<3$, asymptotic flat directions can open up along vector multiplets, but these do not cause wall-crossing

wall-crossing occurs only if an asymptotic flat direction opens up along the scalar multiplet

# Chern-Simons theories and dualities

how do we deal with Chern-Simons term
in this inherently 0-dimensional computation?

$$\frac{\kappa}{4\pi}\int \mathrm{tr}\,(A\wedge dA+\cdots)$$

supersymmetric Landau problem
on the spatial holonomy torus !

$$\frac{\kappa}{4\pi}\int dt\ \mathrm{tr}\,(u_1\dot{u}_2-u_2\dot{u}_1+\cdots)$$

which leads to a polynomial with degree = rank

$$\lim_{\beta\to 0} \Omega^{3d} = \Omega_0 + \kappa^1 \cdot \Omega_1 + \cdots + \kappa^r \cdot \Omega_r$$

$$\Omega_0 \equiv \lim_{\beta\to 0} \Omega^{3d}_{\kappa=0} = \sum_{\Theta_*}^{3d} \sum_{I\in\mathcal{N}_*} \mathcal{Z}_I^{\Theta_*}$$

the obvious uplift of $d$=1 index to $d$=3,
with the proliferation of the holonomy saddles

$$\Omega_0^{\mathrm{WCP}} = \left( \begin{array}{cc} -\sum_I q_I^2 & \xi > 0 \\ 0 & \xi_I < 0 \end{array} \right.$$

# for $SO(2)$ Chern-Simons theories

$$\lim_{\beta\to 0}\Omega^{3d} = \Omega_0 + \kappa\cdot\Omega_1$$

$$\Omega_1 = \frac{1}{(2\pi)^{|\mathcal{S}|/2}}\int d^{|\mathcal{S}|}s\int\prod_J d(y_J^2/2)\; e^{-\left(\sum_{\mathcal{S}} s^2+\sum_J y_J^2\right)/2}$$

again, all the integral reduces to ordinary Gaussian ones, and all the relevant details are stored into the integration domain

$$W = W(P;X_J^2)$$

$$s\equiv\partial_P W$$

$$y_J\equiv X^J\sqrt{q_J\vec{u}^2+(W_J)^2}$$

## for $SO(2)$ Chern-Simons theories

$$\lim_{\beta \to 0} \Omega^{3d} = \Omega_0 + \kappa \cdot \Omega_1$$

$$\Omega_1 = \frac{1}{(2\pi)^{|\mathcal{S}|/2}} \int d^{|\mathcal{S}|} s \int \prod_K d(y_K^2/2)\; e^{-\left(\sum_{\mathcal{S}} s^2 + \sum_K y_K^2\right)/2}$$

the holonomy variables decoupled from the integrand and contribute via an overall volume factor, which is essential in relating the integer $\kappa$ to the magnetic "flux" of the Landau problem

$$W = W(P; X_J^2)$$

$$s \equiv \partial_P W$$

$$y_J \equiv X^J \sqrt{q_J \vec{u}^2 + (W_J)^2}$$

hence, no holonomy saddles for this piece

# for $SO(2)$ Chern-Simons theories

$$\lim_{\beta\to 0} \Omega^{3d} = \Omega_0 + \kappa \cdot \Omega_1$$

$$\Omega_1 = \frac{1}{(2\pi)^{|\mathcal{S}|/2}} \int d^{|\mathcal{S}|}s \int \prod_J d(y_J^2/2)\; e^{-\left(\sum_{\mathcal{S}} s^2 + \sum_J y_J^2\right)/2}$$

the holonomy variables decoupled from the integrand and contribute via an overall volume factor, which is essential in relating the integer $\kappa$ to the magnetic "flux" of the Landau problem

$$W = W(P; X_J^2)$$

$$s \equiv \partial_P W$$

$$y_J \equiv X^J \sqrt{q_J \vec{u}^2 + (W_J)^2}$$

hence, no holonomy saddles for this piece

# for $SO(2)$ Chern-Simons theories

$$\lim_{\beta \to 0} \Omega^{3d} = \Omega_0 + \kappa \cdot \Omega_1$$

$$\Omega_1 = \frac{1}{(2\pi)^{|\mathcal{S}|/2}} \int d^{|\mathcal{S}|} s \int \prod_J d(y_J^2/2)\; e^{-(\sum_{\mathcal{S}} s^2 + \sum_J y_J^2)/2}$$

$$= \frac{1}{(2\pi)^{|\mathcal{S}|/2}} \int d^{|\mathcal{S}|} s\; e^{-(\sum_{\mathcal{S}} s^2)/2}$$

$$W = W(P; \cancel{X_J^2})$$

$$s \equiv \partial_P W$$

$$= \lim_{\beta \to 0} \Omega^{0d}_{\text{neutral scalars only}}$$

# for $SO(2)$ Chern-Simons theories

$$\lim_{\beta \to 0} \Omega^{3d} = \Omega_0 + \kappa \cdot \Omega_1$$

winding number of
charge-neutral sector

$d$=1 winding numbers
amplified by holonomy saddles

wall-crossing occurs at the level of the individual Gaussian integrals that construct

$$\Omega_0,\ \Omega_1,\ \Omega_2,\ \cdots$$

and thus occurs in all dimensions including d=3

# a WCP-like prototype

$$W = P \cdot \left( \sum_J q_J X_J^2 - \xi \right) + \frac{1}{2} \left( mP^2 + \alpha P|P| \right)$$

$$\mathcal{I}^{3d} = \begin{cases} \kappa - \frac{1}{2} \sum_I q_I^2 & m > |\alpha| \\ -\sum_{q_J > 0} q_J^2 & -|\alpha| < m < |\alpha|, \ \alpha < 0 \\ -\sum_{q_J < 0} q_J^2 & -|\alpha| < m < |\alpha|, \ \alpha > 0 \\ -\kappa - \frac{1}{2} \sum_I q_I^2 & m < -|\alpha| \end{cases}$$

which, with twice the supersymmetries, reduces to

$$W = \left( \Sigma \cdot \left( \sum_J q_J X_J^2 - \xi \right) - \frac{1}{2}\kappa \Sigma^2 \right)$$

$P \to \Sigma$

$m \to -\kappa$

$\alpha \to 0$

$$\mathcal{I}^{3d} = \begin{cases} \kappa - \frac{1}{2}\sum_I q_I^2 & \kappa < 0 \\ -\kappa - \frac{1}{2}\sum_I q_I^2 & \kappa > 0 \end{cases}$$

$$= -|\kappa| - \frac{1}{2}\sum_J q_J^2$$

no wall-crossing but vacuum locations depend on sign of $\xi$

reproduces Intriligator-Seiberg counting
for d=3 N=2 Chern-Simons-Matter,
and shows systematically
how wall-crossing turns off there

# some $d=3$ $N=1$ dual pairs

Benini, Bienvenuti 2018

| $SO(2)_{\frac{1}{2}}$ | | Wess-Zumino |
| --- | --- | --- |
| | $\mathcal{I}^{3d} = 0$ | |
| $W = \frac{m}{2}\vert Q\vert^2 - \frac{1}{4}\vert Q\vert^4$ | | $W = P(\vert X\vert^2 + m) - \frac{1}{3}P^3$ |

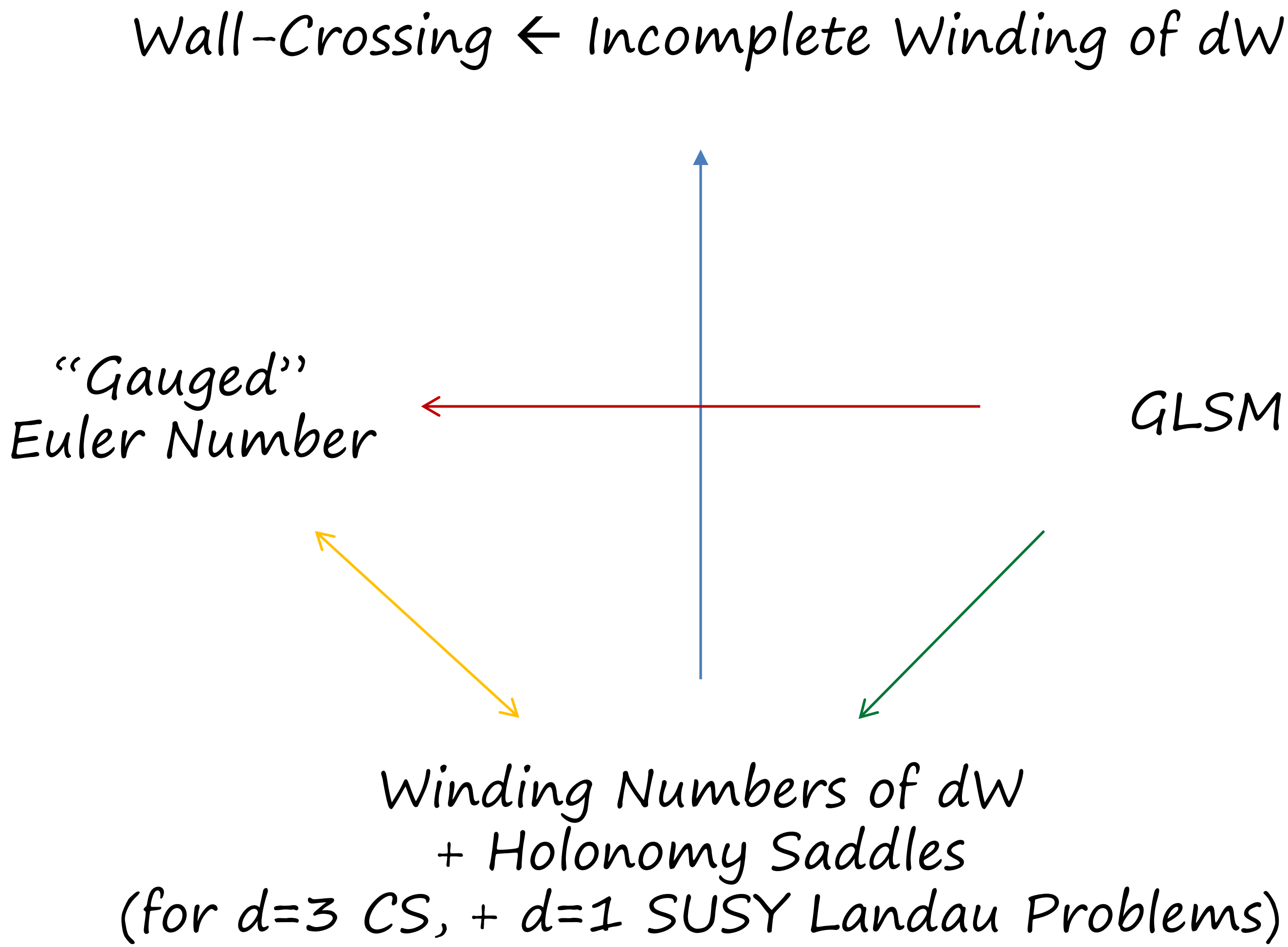
| $SO(2)_{-\frac{1}{2}}$ | | $SO(2)_{\frac{1}{2}}$ |
| --- | --- | --- |
| | $\mathcal{I}^{3d} = -1$ | |
| $W = \frac{m}{2}\vert Q\vert^2 - \frac{1}{4}\vert Q\vert^4$ | | $W = P(\vert X\vert^2 + m) - \frac{1}{3}P^3$ |

# some $d=3$ $N=1$ dual pairs

Benini, Bienvenuti 2018

| Wess-Zumino | | $SO(2)_{-\frac{1}{2}}$ |
|---|---|---|
| $W=\frac{m}{2}\lvert Q\rvert^2$ | $\mathcal{I}^{3d}=1$ | $W=P(\lvert X\rvert^2+m)+\frac{1}{2}P^2$ |

| $SO(2)_{\frac{3}{2}}$ | | $SO(2)_{-\frac{3}{2}}$ |
|---|---|---|
| $W=\frac{m}{2}\lvert Q\rvert^2-\frac{1}{4}\lvert Q\rvert^4$ | $\mathcal{I}^{3d}=-1$ | $W=P(\lvert X\rvert^2+m)-\frac{1}{3}P^3$ |


Wall-Crossing ← Incomplete Winding of dW
"Gauged"
Euler Number
GLSM
Winding Numbers of dW
+ Holonomy Saddles
(for d=3 CS, + d=1 SUSY Landau Problems)