Gromov-Witten Theory on Elliptic 4-folds

&

Weakness of Gravity in 4 Dimensions

Based mainly on

2008.xxxxx	w/ Daniel Klaewer, Timo Weigand, Max Wiesner
2005.10837	w/ Wolfgang Lerche, Guli Lockhart, Timo Weigand
1901.08065 , 1808.05958	w/ Wolfgang Lerche, Timo Weigand

Seung-Joo Lee (IBS-CTPU)

GLSM—2020@Virginia Tech/Online

20-08-20

VIRGINIA TECH.

Institute for Basic Science

Gromov-Witten Theory on Elliptic 4-folds

&

Weakness of Gravity in 4 Dimensions

Based mainly on

2008.xxxxx w/ Daniel Klaewer, Timo Weigand, Max Wiesner

2005.10837 w/ Wolfgang Lerche, Guli Lockhart, Timo Weigand

1901.08065, 1808.05958 w/ Wolfgang Lerche, Timo Weigand

4-folds
4D N=1 Phys.

3-folds
6D N=(1,0) Phys.

Seung-Joo Lee (IBS-CTPU)

GLSM—2020@Virginia Tech/Online

20-08-20

VIRGINIA TECH.

Institute for Basic Science

Motivation

Question 1: Gromov-Witten Theory

- **Modularity of GW Invariants on Elliptic 4-folds**
  - Elliptic Calabi-Yau manifolds with non-trivial sections:

    $\pi : Y_{d+1} \to B_d$ (d=3 for 4-folds) with $\mathrm{rk} MW(\pi) > 0$ (*rk-1 mostly assumed in this talk*)
  - Generating functions for appropriately collected GW invariants:

$$\mathcal{F}^{Y_4}_{C_{\mathrm{b}};G_4}(q,\xi) = \sum_{n,r} \langle G_4 \rangle^{Y_4}_{C_{\mathrm{b}}+nC_{\mathrm{E}}+rC_{\mathrm{f}}}\, q^n \xi^r$$

  - Such generating functions are expected to exhibit modularity

— 3-folds: a lot known, Jacobi [Haghighat, Murthy, Vafa, Vandoren '15], [Huang, Katz, Klemm '15]

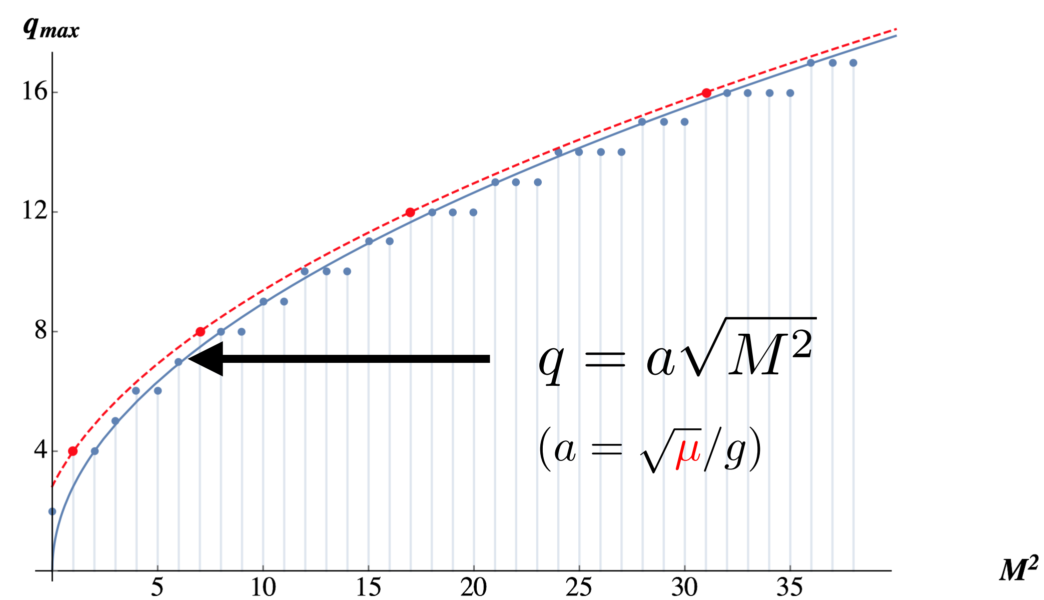
— 4-folds: much less known, not Jacobi in general [S.-J.L., Lerche, Weigand '19]

◆ Characterization of a behavior under modular/elliptic transformations?

Motivation

Question 2: Weak Gravity Conjectures

- **Weak Gravity Conjecture(s)**
  - Gravity is "weakest" [Arkani-Hamed, Motl, Nicolis, Vafa '06]
    - $\exists$ a super-extremal particle with $g^2 q^2 \geq \mu M^2$
    - $\exists$ a super-extremal *tower*
    - $\exists$ a super-extremal *sublattice*
  - Minimal ver. $\Leftarrow$ ***Tower*** ver. $\Leftarrow$ ***Sublattice*** ver.
  - Verified in F-theory/heterotic compactifications (in weak gauge-coupling regimes) for



— ***General*** 6d N=(1,0) EFTs [S.-J.L., Lerche, Weigand '18]

— ***Special*** (non-generic) 4d N=1 EFTs [S.-J.L., Lerche, Weigand '19]

◆ Verification for ***general*** EFTs with 4 supercharges?

Main Results

A Schematic Summary

- **Modularity of GW Invariants on Elliptic 4-folds**
  - The generating functions, $\mathcal{F}^{Y_4}_{C_{\rm b};G_4}(q,\xi)$, once shifted as $\mathcal{Z} = -q^{E_0}\mathcal{F}$, take the form:

$$\mathcal{Z} = \boxed{Z_{-1,m}} + \xi\partial_\xi \boxed{Z_{-2,m}}$$

Jacobi forms of weight w (= -1, -2) and index m

to be determined by the topology

  - Connection b/w GW theory on an elliptic CY 4-fold and those on 3-folds therein

- **Weak Gravity Conjecture(s)**
  - The conjectures hold for general 4d N=1 compactifications of F-theory with fluxes
  - There exist a tower of super-extremal particles, which form a (shifted) sublattice:

— minimal, tower, and sublattice versions hold

Why Care

Where are **GLSMs**?

- **Modular Structure of GW invariants**
  - **GLSM** lovers also love GW invariants…
  - An analytic formula can be found for $\mathcal{F}_{C_{\mathrm{b}};G_4}(q,\xi)$ if 4-folds admit a **GLSM** description
- **Elliptic Genera in 4d N=1 Theories**
  - $\mathcal{Z} = -q^{E_0}\mathcal{F}$ are in fact elliptic genera of certain solitonic strings with N=(0,2) SUSY

  — critical heterotic strings (perturbative or not)

  — non-critical strings (such as "E-strings")
  - One can gain direct access to the elliptic genera via a UV **GLSM** (if exists)

Outline

Solitonic Strings in 4d F-theory

1. Arena

   Rudiments of F-theory

Part I. Gromov-Witten Theory on Elliptic 4-folds

2. Elliptic Genera

   Connection to the GW Theory and Modularity

3. Example

   An Explicit Model with a U(1) Vector

Part II. Weak Gravity Conjectures in 4 Dimensions

4. Tensionless String

   Geometry of Weak-Coupling Limits

5. Super-Extremal Tower

   Quasi-Jacobi Elliptic Genera

Conclusions

Solitonic Strings in 4d F-theory

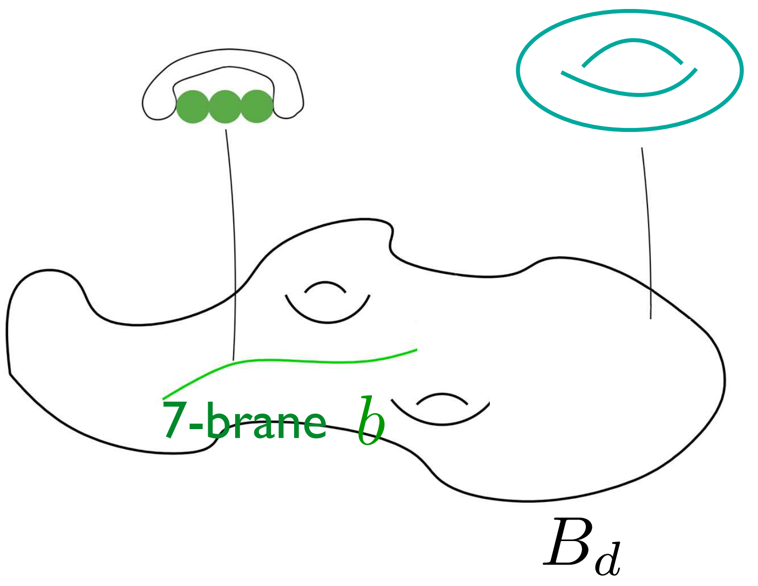
1/5. Arena

F-theory

Elliptic Fibration and Gauge Algebra

- **D-dim'l F–theory**
  - IIB string theory on B_d with varying axio-dilaton

    — d = 2, 3 lead to eff. physics in $D = 10 - 2d = 6, 4$ dimensions
  - Dilaton profile via an elliptic CY manifold, $\pi : Y_{d+1} \to B_d$
  - Gauge fields from 7-branes on a divisor ***b***



- **Gravity and Gauge Dynamics**
  - Planck scale and gauge coupling are given as

$$M_{\rm Pl}^{D\text{-}2} \sim \mathrm{vol}_J(B_d)$$
$$1/g^2 \sim \mathrm{vol}_J(b)$$

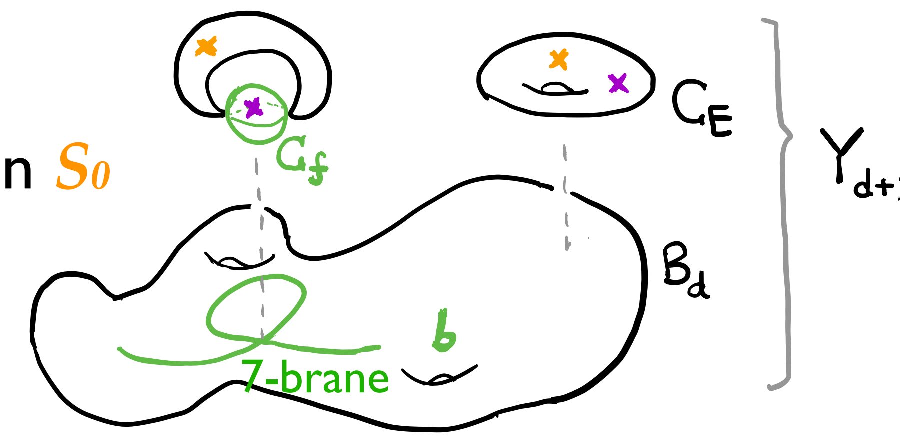
  - In this talk: mostly assume $G=U(1)$ to be concrete

- **Moduli Space**
  - The Kahler class J can be deformed to change the couplings

Geometric Origin of U(1)s

Sections and Shioda Map

- **Sections and Shioda Map**



  - Non-zero (rational) section S in add. to the zero section S_0
  - Shioda map (see e.g. the reviews [Weigand '18], [Cvetic, Lin '18])

$$\boxed{\sigma(S) = S - S_0 - \pi^*(\mathfrak{D})}$$

w/ the def. properties: $\sigma \circ \pi^* w_6 = 0\,, \qquad \sigma \circ S_0 \circ \pi^* w_4 = 0 \qquad \forall\, w_6 \in H^6(B_3)\,, \quad w_4 \in H^4(B_3)$

- **U(1) Vectors**
  - F/M-duality leads to a U(1) vector: $C_3 = A \wedge \sigma(S) + \cdots$
  - The gauge coupling is computed as $1/g^2 = \int_{Y_4} \sigma \wedge *\sigma \;\longrightarrow\; \mathrm{vol}(b)$

  where $b := -\pi_*(\sigma \circ \sigma) \in H^2(B_3)$ is the height-pairing

  - Fibral curve **C_f** (in addition to the full fiber **C_E**) leads to charged matter

$$(S_0, \sigma(S)) \circ C_{\mathrm{E}} = (1, 0)$$

$$(S_0, \sigma(S)) \circ C_{\mathrm{f}} = (0, 1)$$

Fluxes

Vertical and Transversal

- **Vertical Fluxes**

  from $H^{4,0}(Y_4)$ via variation of Hodge structure

  $H^{1,1}(Y_4) \wedge H^{1,1}(Y_4)$

  - Supersymmetric fluxes $G_4 \in H^{2,2}(Y_4) = H^{2,2}_{\rm hor}(Y_4) \oplus H^{2,2}_{\rm vert}(Y_4) \oplus H^{2,2}_{\rm rem}(Y_4)$ [Greene, Morrison, Plesser '94] [Braun, Watari '14]
  - Genus-0 GW invariants via mirror symmetry of 4-folds:

  IIA on *X4* with $G_4^X \in H^{2,2}_{\rm hor}(X_4)$ ◀··················▶ IIA on *Y4* with $G_4^Y \in H^{2,2}_{\rm vert}(Y_4)$

- **4d F-theory on *Y4***
  - Lorenz inv. imposes the fluxes to lie in the transversal subspace $H^{2,2}_{(-1)}(Y_4, \mathbb{R})$:

$$\boxed{\begin{aligned} G_4 \circ \pi^* D^\alpha \circ \pi^* D^\beta &= 0 \\ G_4 \circ S_0 \circ \pi^* D^\alpha &= 0 \end{aligned}} \qquad \forall D^\alpha \in H^{1,1}(B_3)$$

  - The vertical flux space decompose in general as

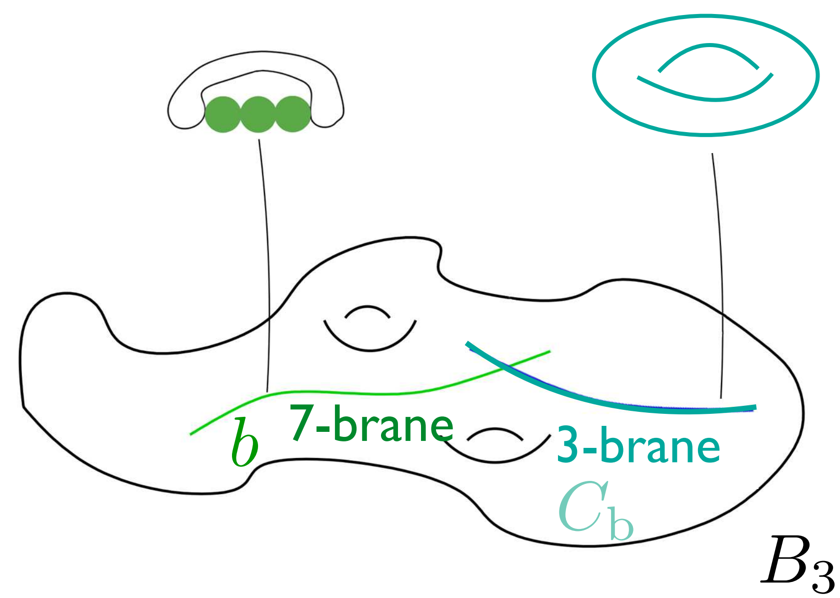
$$H^{2,2}_{\rm vert}(Y_4,\mathbb{R}) = \underbrace{H^{2,2}_{(0)}(Y_4,\mathbb{R})}_{\mathrm{Span}\langle (S_0 + \frac{1}{2}\pi^* \bar{K}_{B_3}) \wedge \pi^* D^\alpha \rangle} \oplus \underbrace{H^{2,2}_{(-1)}(Y_4,\mathbb{R})}_{\mathrm{Span}\langle \sigma(S) \wedge \pi^* D^\alpha \rangle} \oplus \underbrace{H^{2,2}_{(-2)}(Y_4,\mathbb{R})}_{\mathrm{Span}\langle \pi^* D^\alpha \wedge \pi^* D^\beta \rangle}$$

Transversal U(1) Fluxes

Rudiments

Solitonic Strings

- **Branes on Curves**
  - D7 brane on b : gauge multiplet
  - D3 brane on $C_{\rm b}$: effective string



- **The String WS**
  - Twisted reduction along $C_{\rm b}$ of N=4 SYM with varying coupling → N=(0,2) WS of a 4d string

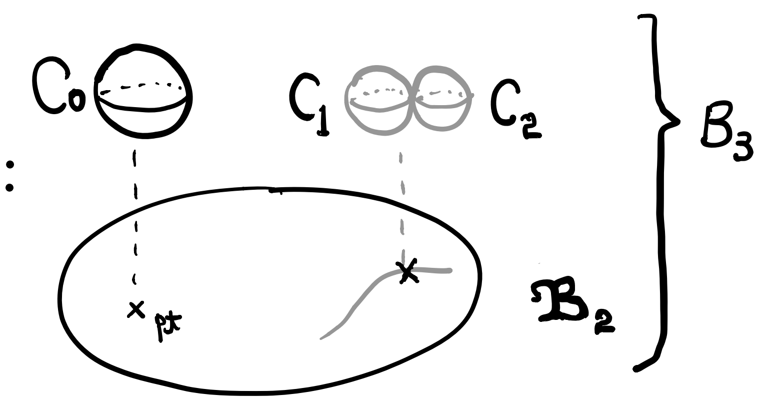
[Martucci '14], [Haghighat, Murthy, Vafa, Vandoren '15], [Lawrie, Schafer-Nameki, Weigand '16]

  - 3-7 modes present at $C_{\rm b} \cdot b$ are charged under the 4d gauge group

Rudiments

Solitonic Strings: Critical or Non-Critical

- **Fibered Base 3-fold**
  - Will mostly consider base B_3*'s* admitting a rational fibration:
    $p : B_3 \to \mathfrak{B}_2 \quad , \qquad C_0 := p^{-1}(pt)$



- **D3 on $C_b = C_0$**
  - 4d critical heterotic string
  - Normal bundle $\mathcal{O} \oplus \mathcal{O}$
  - #(charged Fermi fields in the 3-7 sector) $= \; 8\bar{K}_{B_3} \cdot C_0 = 16$ ⟶ $E_8 \times E_8$ or $SO(32)$

- **D3 on $C_b = C_a$ (a=1,2)**
  - 4d non-critical "E-string"
  - Normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}$
  - #(charged Fermi fields in the 3-7 sector) $= \; 8\bar{K}_{B_3} \cdot C_0 = 8$ ⟶ E_8

Part I.

Gromov-Witten Theory on Elliptic 4-folds

2/5. Elliptic Genera

Elliptic Genera

U(1)-Refined

- **Elliptic Genus**
  - $\mathcal{Z}(\tau, z) \equiv \mathrm{Tr}_{\mathrm{RR}}\left[(-1)^F F\, q^{H_L} \bar{q}^{H_R} \xi^J\right]$, where $q = e^{2\pi i \tau}$ and $\xi = e^{2\pi i z}$
  - Refined by U(1) flavor fugacity: especially crucial in 4d as $\mathcal{Z}(\tau, z=0) = 0$
  - Information on the left-movers of the N=(0,2) worldsheet CFT
  - For the heterotic case: level-matched to give particle excitations

- **In the Current Context**
  - Expansion coefficients are (related to) the GW invariants
  - Well-defined behavior under modular/elliptic transformations

Connection to GW Theory

6d Strings

- **F/M Duality**

6d F-theory on S^1 [Klemm, Mayr, Vafa '96] *5d M-theory*

- String wrapped on the S^1 with winding 1 and momentum n
- Elliptic genus of 6d string

- M2 wrapped on $C_{\mathrm{b}} + n\, C_{\mathrm{E}}$
- BPS invariants for the 5d particles

$$\mathcal{Z}^{Y_3}_{C_{\mathrm{b}}}(\tau, z) = -q^{E_0} \sum N(n,r) q^n \xi^r, \quad \text{with } E_0 = -\frac{1}{2} C_{\mathrm{b}} \cdot \bar{K}_{B_2}$$

- Expansion coefficients $N(n,r)$
- (τ, z)

- Genus-0 GW invariants: $\langle \cdot \rangle^{Y_3}_{C_{n,r}}$ for $C_{n,r} = C_{\mathrm{b}} + n\, C_{\mathrm{E}} + r\, C_{\mathrm{f}}$
- Kahler parameters for $(C_{\mathrm{E}}, C_{\mathrm{f}})$

- **GW Invariants on Elliptic 3-folds**
  - $\mathcal{F}^{Y_3}_{C_{\mathrm{b}}}(q, \xi) = \sum_{n,r} \langle \cdot \rangle^{Y_3}_{C_{n,r}} q^n \xi^r$, computable via mirror symmetry for 3-folds
  - $\mathcal{Z}^{Y_3}_{C_{\mathrm{b}}}(q, \xi) = -q^{E_0} \mathcal{F}^{Y_3}_{C_{\mathrm{b}}}(q, \xi)$

[Klemm, Mayr, Vafa '96],
[Haghighat, Iqbal, Kozcaz, Lockhart, Vafa '13],
[Haghighat, Klemm, Lockhart, Vafa '14] ...

Connection to GW Theory

4d Strings

- **Proposal** (analogy with 3-fold cases)

  For the string associated w/ a primitive base curve C_b, the $N(n,r)$ is the genus-0 GW invariant for the curve $C_{n,r} = C_\mathrm{b} + n\, C_E + r\, C_\mathrm{f}$ w.r.t. the flux $G \in H^{2,2}(Y_4)$:

$$N(n,r) = \langle G \rangle^{Y_4}_{0,C_{n,r}} \ (=: \langle G \rangle^{Y_4}_{C_{n,r}})$$

- **GW Invariants on Elliptic 4-folds with Fluxes**

  - $\mathcal{F}^{Y_4}_{C_\mathrm{b};G}(q,\xi) = \sum_{n,r} \langle G \rangle^{Y_4}_{C_{n,r}}\, q^n \xi^r$, computable via mirror symmetry for 4-folds
  - $\mathcal{Z}^{Y_4}_{C_\mathrm{b};G}(q,\xi) = -q^{E_0} \mathcal{F}^{Y_4}_{C_\mathrm{b};G}(q,\xi)$, with $E_0 = -\frac{1}{2} C_\mathrm{b} \cdot \bar{K}_{B_3}$

  [Greene, Morrison, Plesser '94], [Mayr '97], [Klemm, Yau '98], [Haghighat, Movasati, Yau '15], [Cota, Klemm, Schimannek '17]

- **Remarks**

  for Calabi-Yau

  for d=3, g=0 and n=1

  - $\langle A_1, \cdots, A_n \rangle^{Y_{d+1}}_{g,C}$ count stable maps with n pts fixed, whose images lie in $A_i \in H_*(Y_{d+1})$
  - Stable-map moduli space has $\dim_\mathrm{v} \mathcal{M}_{g,n}(Y_{d+1}, C) = (d-2)(1-g) + n = 2$
  - Natural 4-fold invariants: $\langle G \rangle^{Y_4}_C = [\overline{\mathcal{M}}_{g,n}(Y_4, C)]^{\mathrm{vir}}\, \mathrm{ev}^* G$, $\quad G \in H^4(Y_4)$

Modularity

6d Strings

- **Jacobi** **[almost]**
  - The elliptic genus $\mathcal{Z}^{Y_3}_{C_\mathrm{b}}$ of a 6d string is a (mero. weak) Jacobi form [mod E_2's]

    — Modular/elliptic transform: $$\varphi_{w,m}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^{w}e^{2\pi i\frac{mc}{c\tau+d}\frac{z^2}{2}}\varphi_{w,m}(\tau,z)$$
    $$\varphi_{w,m}(\tau,z+\lambda\tau+\mu)=e^{-2\pi i m(\frac{\lambda^2}{2}\tau+\lambda z)}\varphi_{w,m}(\tau,z)$$

    — Ring of Jacobi forms: $\mathcal{J}_{*,*}=\oplus\mathcal{J}_{w,m}=\mathbb{C}[E_4,E_6,\phi_{0,1},\phi_{-2,1},\phi_{-1,2}]$

  - Expected weight and index are

    — $w=-\dfrac{D-2}{2}=-2$ (true e.g. of pert. het. string in D=6 dim) [Schellekens, Warner '86] ...

    — $m=\dfrac{1}{2}b\cdot C_\mathrm{b}$, where b is the U(1) height-pairing and C_b is the base curve [S.-J.L., Lerche, Weigand '18]

- **Appearance of E_2's**
  - Quasi-modularity if C_b splits: e.g., $C_0=C_1+C_2$ [Alim, Scheidegger '12], [Klemm, Manschot, Wotschke '12] ...
  - Modular vs. holomorphic anomalies: E_2 vs. $\hat{E}_2:=E_2-\dfrac{3}{\pi\mathrm{Im}(\tau)}$

Modularity

4d Strings

- **Jacobi [almost] ?**
  - The elliptic genus $\mathcal{Z}^{Y_4}_{C_{\mathrm{b}};G}$ of a 4d string is ***not*** Jacobi [even mod E_2's]
  - Expected would-be weight and index are

— $w = -\frac{D-2}{2} = -1$

— $m = \frac{1}{2} b \cdot C_{\mathrm{b}}$, where b is the height-pairing and C_b is the base curve

- **Appearance of a "Derivative Sector"**
  - **Prop.** $\mathcal{Z}^{Y_4}_{C_{\mathrm{b}};G}(q,\xi) = Z_{-1,m}(q,\xi) + \xi\partial_\xi Z_{-2,m}(q,\xi)$ [S.-J.L., Lerche, Lockhart, Weigand '20]

$$Z_{-1,m}(q,\xi) = \frac{\Phi^{\mathrm{QM}}_{w,m}(q,\xi)}{\eta(q)^{12C_{\mathrm{b}}\cdot\bar{K}_{B_3}}}$$

$$Z_{-2,m}(q,\xi) = \frac{\Phi^{\mathrm{QM}}_{w-1,m}(q,\xi)}{\eta(q)^{12C_{\mathrm{b}}\cdot\bar{K}_{B_3}}}$$

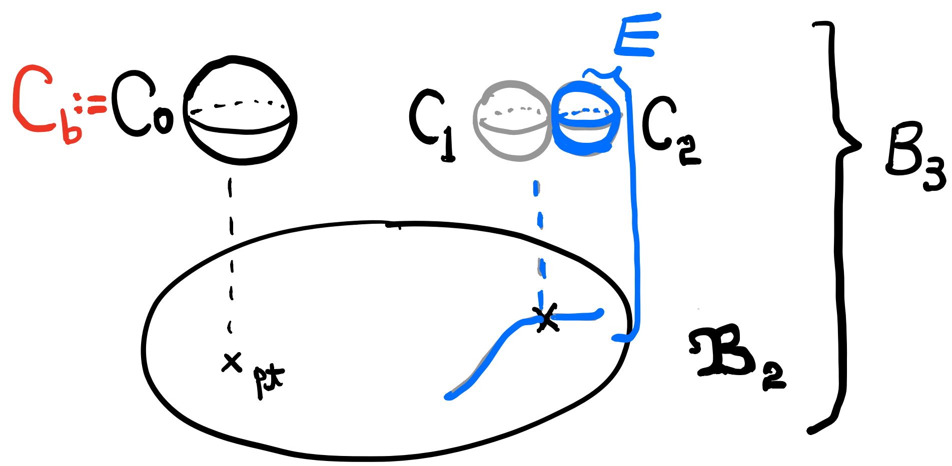
$$w = -1 + 6C_{\mathrm{b}}\cdot\bar{K}_{B_3}$$

$$m = \frac{1}{2} b \cdot C_{\mathrm{b}}$$

Derivative Sector

4d Heterotic String

- **Fibered Base 3-fold**
  - A rational fibration p, with the fibers degenerating along a single curve in $\mathfrak{B}_2$

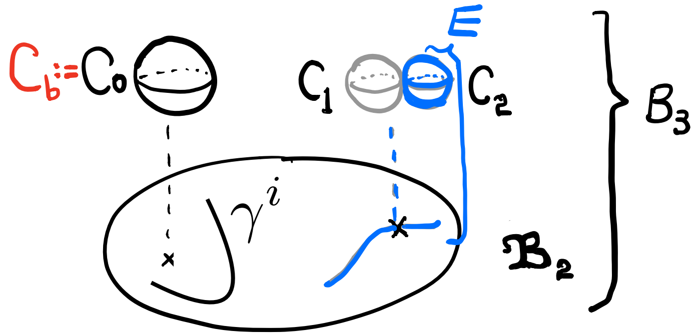


  - Blowup divisor E and exceptional section S_-

- **Flux Basis**
  - $\boxed{G^{\alpha}_{U(1)} = \sigma \wedge \pi^* D^{\alpha} \,, \quad \alpha = 1, \cdots, h^{1,1}(B_3)}$

    where the divisor basis can be taken, e.g., as $D^{\alpha} \in \{S_-, E, p^* \gamma^i\}$

$$H^{1,1}(\mathfrak{B}_2) = \mathrm{Span}\, \langle \gamma^i \rangle_{i=1,\cdots,h^{1,1}(\mathfrak{B}_2)}$$



Derivative Sector

Geometric Origin [S.-J.L., Lerche, Lockhart, Weigand '20]

Study $\langle G^{\alpha}\rangle^{Y_4}_{C_{n,r}}$ for curve classes $C_{n,r} = C_0 + n\,C_E + r\,C_f$
where $G^{\alpha} = \sigma\wedge\pi^* D^{\alpha}, \quad D^{\alpha}\in\{S_-, E, p^*\gamma^i\}$

- **Special Fluxes**

  - For $G^{\alpha=i} = \sigma\wedge\pi^* p^*\gamma^i$:

    Divisor equation

    $\langle G^i\rangle^{Y_4}_{C_{n,r}} = \langle\sigma\rangle^{Y_3^i := \pi^* p^*\gamma^i}_{C_{n,r}} = (\sigma\cdot C_{n,r})\,\langle\,\cdot\,\rangle^{Y_3^i}_{C_{n,r}} = \boxed{r}\,\langle\,\cdot\,\rangle^{Y_3^i}_{C_{n,r}} \Rightarrow \mathcal{Z}^{Y_4}_{C_0;\,c_i G^i}(q,\xi) = \boxed{\xi\partial_\xi}\,\mathcal{Z}^{Y_3 := \pi^* p^*(c_i\gamma^i)}_{C_0}(q,\xi)$

  - For $G_{\rm QM} = \sigma\wedge\pi^* D$ with $D\cdot b\cdot p^*\gamma = 0\ \ \forall\gamma\in H^{1,1}(\mathcal{B}_2)$ (dubbed as ***Quasi-Modular fluxes***) :

    $\mathcal{Z}^{Y_4}_{C_0;\,G_{\rm QM}}(q,\xi)$ is a Jacobi form of $w = -1$ and $m = \frac{1}{2}b\cdot C_{\rm b}$ [S.-J.L., Lerche, Weigand '19]

- **General Fluxes**

  - $\mathcal{Z}_{C_{\rm b};G}(q,\xi) = Z_{-1,m}(q,\xi) + \xi\partial_\xi Z_{-2,m}(q,\xi)$
  - Used $\mathcal{Z}^{Y_3}_{C_0}(q,\xi)$ is a Jacobi form of $w = -2$ and $m = \frac{1}{2}b\cdot C_{\rm b}$

  — 1) Physically clear if the hyper surface Y_3 is Calabi-Yau

  — 2) Observed $Z_{-2,m}(q,\xi) = \mathcal{Z}_{C_0;\,G_{(-2)}}(q,\xi)$ for $G_{(-2)} = \pi^* D\wedge\pi^* b$ (cf.) [Oberdieck, Pixton '17]

  - Manifestation of the quasi-Jacobi nature (cf.) [Oberdieck, Pixton '17]

Quasi-Jacobi Forms

Definition and Examples

- **Definition**
  - An almost holo. function:

$$\Phi(\tau,z)=\sum_{i,j\geq 0}\varphi_{i,j}(\tau,z)\left(\frac{1}{\mathrm{Im}\tau}\right)^i\left(\frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)^j$$

  - If $\Phi(\tau,z)$ transforms as a Jacobi form, then $\varphi_{0,0}(\tau,z)$ is called quasi-Jacobi

- **Examples**
  - $E_2(\tau)$ is a quasi-Jacobi form:

$$\hat{E}_2(\tau):=E_2(\tau)-\frac{3}{\pi\mathrm{Im}\tau}$$

  - $\xi\partial_\xi\varphi_{w,m}(\tau,z)$ is a quasi-Jacobi form:

$$(\xi\partial_\xi+4\pi i m\frac{\mathrm{Im}z}{\mathrm{Im}\tau})\varphi_{w,m}(\tau,z)=\varphi_{w+1,m}(\tau,z)$$

$\Rightarrow$ 4d N=(0,2) elliptic genera are a (mero.) quasi-Jacobi form!

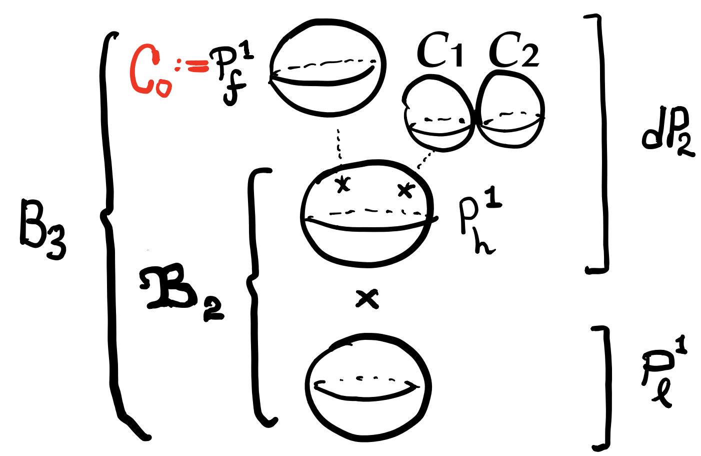
3/5. Example

[S.-J.L., Lerche, Lockhart, Weigand ’20]

The 3-Base and the Elliptic Fibration

$\mathrm{Bl}_1\, P^2_{112}[4]$ **Fibration on** $dP_2 \times P^1_\ell$

- **Base,** $B_3 = dP_2 \times P^1_\ell$



- **Rational Fibration**
  - dP_2 viewed as $\mathrm{Bl}_1\, \mathbf{F}_1$, where $\mathbf{F}_1$ is a fibration of P^1_f (=: C_0) over P^1_h
  - $p : B_3 \to \mathfrak{B}_2$ with $\mathfrak{B}_2 = P^1_{\mathrm{h}} \times P^1_\ell$

- **Cohomology Basis**
  - $H^{1,1}(\mathfrak{B}_2) = \mathrm{Span}\,\langle \gamma^1, \gamma^2 \rangle$ with $\gamma^1 = P^1_{\mathrm{h}}\,, \quad \gamma^2 = P^1_\ell$
  - $H^{1,1}(B_3) = \mathrm{Span}\,\langle D^\alpha \rangle$ with $D^1 = p^*\gamma^2\,, \quad D^2 = S_-\,, \quad D^3 = p^*\gamma^1\,, \quad D^4 = p^*\gamma^2 + S_- - E$

    where $S_- \simeq P^1_{\mathrm{h}} \times P^1_\ell$ is the exceptional section and $E \simeq C_2 \times P^1_\ell$ is the blowup divisor

The 3-Base and the Elliptic Fibration

$\mathrm{Bl}_1 P^2_{112}[4]$ **Fibration on** $dP_2 \times P^1_\ell$

- **Fiber,** $\mathrm{Bl}_1 P^2_{112}[4]$
  - To realize *rk-1* Mordell-Weil lattice
  - Fibral toric coordinates: u, v, w, s w/ the GLSM charges,

u	v	w	s
1	1	2	0
0	1	1	1

- **Fibration** [Morrison, Park '12]
  - Take a hypersurface of the form:

$$\hat{P} = sw^2 + b_0 s^2 u^2 w + b_1 suvw + b_2 v^2 w + c_0 s^3 u^4 + c_1 s^2 u^3 v + c_2 su^2 v^2 + c_3 uv^3$$

$$[b_0] = \beta\,, \qquad [b_1] = \bar{K}\,, \qquad [b_2] = 2\bar{K} - \beta\,, \qquad [c_0] = 2\beta\,,$$
$$[c_1] = \beta + \bar{K}\,, \qquad [c_2] = 2\bar{K}\,, \qquad [c_3] = 3\bar{K} - \beta\,, \qquad [c_4] = 4\bar{K} - 2\beta\,.$$

  - Take $\beta = 2\bar{K}_{B_3}$
  - Height pairing, $b := -\pi_*(\sigma \circ \sigma) = 6\bar{K}_{B_3} - 2\beta = 2\bar{K}_{B_3}$

Computation of the Elliptic Genera

4/6d Heterotic Strings

- **Flux Basis**

$G^\alpha = \sigma \wedge \pi^* D^\alpha, \quad \alpha = 1, \cdots, 4\,(= h^{1,1}(B_3))$ ⇒ Most general flux $G = c_\alpha G^\alpha$

- **Low-Degree Invariants = Expansion of the Elliptic Genera**

  - 4d heterotic string ($\mathcal{F}$ on Y_4):

$$\begin{aligned}
\mathcal{Z}^{Y_4}_{C_0;G} \;=\; & -[96c_1\xi^{\pm\bar{1}} + 48c_2\xi^{\pm\bar{1}} + 84c_3\xi^{\pm\bar{1}} + 96c_4\xi^{\pm\bar{1}}] \\
& - q\ [c_1(69280\xi^{\pm\bar{1}} + 20384\xi^{\pm\bar{2}} + 288\xi^{\pm\bar{3}} - 8\xi^{\pm\bar{4}}) \\
& \quad +c_2(99552\xi^{\pm\bar{1}} + 29088\xi^{\pm\bar{2}} + 480\xi^{\pm\bar{3}} - 12\xi^{\pm\bar{4}}) \\
& \quad +c_3(65164\xi^{\pm\bar{1}} + 18896\xi^{\pm\bar{2}} + 252\xi^{\pm\bar{3}} - 8\xi^{\pm\bar{4}}) \\
& \quad +c_4(134192\xi^{\pm\bar{1}} + 39280\xi^{\pm\bar{2}} + 624\xi^{\pm\bar{3}} - 16\xi^{\pm\bar{4}})] \\
& + \mathcal{O}(q^2)\,,
\end{aligned} \tag{B.20}$$

  - 6d heterotic strings ($\mathcal{F}$ on Y_3^i):

$$\begin{aligned}
\mathcal{Z}^{Y_3^1}_{C_0} = 2/q & - \left(252 + 84\xi^{\pm1}\right) - \left(116580 + 65164\xi^{\pm1} + 9448\xi^{\pm2} + 84\xi^{\pm3} - 2\xi^{\pm4}\right) q \\
& - \left(6238536 + 3986964\xi^{\pm1} + 965232\xi^{\pm2} + 65164\xi^{\pm3} + 252\xi^{\pm4}\right) q^2 + \mathcal{O}(q^3)\,,
\end{aligned}$$

$$\mathcal{Z}^{Y_3^2}_{C_0} = 2/q - \left(288 + 96\xi^{\pm1}\right) - \left(123756 + 69280\xi^{\pm1} + 10192\xi^{\pm2} + 96\xi^{\pm3} - 2\xi^{\pm4}\right) q + \mathcal{O}(q^2)$$

Analytic Expression for the Elliptic Genera

4/6d Heterotic Strings

$$
\begin{aligned}
\mathcal{Z}^{Y_4}_{C_0;G} = & -[96c_1\xi^{\pm\bar{1}} + 48c_2\xi^{\pm\bar{1}} + 84c_3\xi^{\pm\bar{1}} + 96c_4\xi^{\pm\bar{1}}] \\
& - q\ [c_1(69280\xi^{\pm\bar{1}} + 20384\xi^{\pm\bar{2}} + 288\xi^{\pm\bar{3}} - 8\xi^{\pm\bar{4}}) \\
& \quad +c_2(99552\xi^{\pm\bar{1}} + 29088\xi^{\pm\bar{2}} + 480\xi^{\pm\bar{3}} - 12\xi^{\pm\bar{4}}) \\
& \quad +c_3(65164\xi^{\pm\bar{1}} + 18896\xi^{\pm\bar{2}} + 252\xi^{\pm\bar{3}} - 8\xi^{\pm\bar{4}}) \\
& \quad +c_4(134192\xi^{\pm\bar{1}} + 39280\xi^{\pm\bar{2}} + 624\xi^{\pm\bar{3}} - 16\xi^{\pm\bar{4}})] \\
& + \mathcal{O}(q^2)\,,
\end{aligned}
\tag{B.20}
$$

$$
= \underbrace{(c_2 + c_4)}_{D\cdot C_0} Z^0_{-1,2} + \underbrace{4(c_2 + c_3 + c_4)}_{D\cdot b\cdot p^*\gamma_1} \overbrace{\frac{1}{4}}^{1/2m} \xi\partial_\xi Z^1_{-2,2} + \underbrace{2(2c_1 + c_2 + 2c_4)}_{D\cdot b\cdot p^*\gamma_2} \overbrace{\frac{1}{4}}^{1/2m} \xi\partial_\xi Z^2_{-2,2}
$$

where

$$
\begin{aligned}
Z^0_{-1,2} &\equiv 84\varphi_{-1,2} \\
Z^1_{-2,2} &\equiv \frac{1}{12}\frac{1}{\eta^{24}}(14E_4E_{6,2} + 10E_{4,2}E_6 - E_{4,1}E_{6,1}) + \frac{1}{12}\frac{1}{\eta^{24}}E_2E^2_{4,1} \\
Z^2_{-2,2} &\equiv \frac{1}{12}\frac{1}{\eta^{24}}(14E_4E_{6,2} + 10E_{4,2}E_6)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Z}^{Y^1_3}_{C_0} = & \ 2/q - (252 + 84\xi^{\pm1}) - (116580 + 65164\xi^{\pm1} + 9448\xi^{\pm2} + 84\xi^{\pm3} - 2\xi^{\pm4})\,q \\
& - (6238536 + 3986964\xi^{\pm1} + 965232\xi^{\pm2} + 65164\xi^{\pm3} + 252\xi^{\pm4})\,q^2 + \mathcal{O}(q^3)\,,
\end{aligned}
\quad = Z^1_{-2,2}
$$

$$
\mathcal{Z}^{Y^2_3}_{C_0} = 2/q - (288 + 96\xi^{\pm1}) - (123756 + 69280\xi^{\pm1} + 10192\xi^{\pm2} + 96\xi^{\pm3} - 2\xi^{\pm4})\,q + \mathcal{O}(q^2) \quad = Z^2_{-2,2}
$$

Part II.

Weak Gravity Conjectures in 4 Dimensions

4/5. Tensionless String

Quantum Gravity Conjectures

String-Theoretic Approach

- **Quantum Gravity Conjectures ~ Swampland Conjectures**
  - Consistency constraints Quantum Gravity (QG) is believed to impose
  - *Landscape* v.s. *Swampland* in the context of string theory

- **Conjectures in the Literature**
  - No Global Symmetry [Banks, Dixon '88]
    Gauge coupling g cannot become zero in presence of gravity
  - Completeness [Polchinski '03]
    The entire charge lattice must be popultated by physical particles
  - Weak Gravity [Arkani-Hamed, Motl, Nicolis, Vafa '06], [Heidenreich, Reece, Rudelius '16-'17], [Montero, Shiu, Soler '16]
    A (sub)lattice/tower of charged particles with $g^2q^2 \geq \mu M^2$ *should exist*
  - Distance Conjecture [Ooguri, Vafa '06]
    An "infinite" deformation of physics leads to a tower of particles with their masses exponentially suppressed

Quantum Gravity Conjectures

String-Theoretic Approach

- **Quantum Gravity Conjectures ~ Swampland Conjectures**
  - Consistency constraints Quantum Gravity (QG) is believed to impose
  - *Landscape* v.s. *Swampland* in the context of string theory

- **Conjectures in the Literature**
  - No Global Symmetry [Banks, Dixon '88]
    Gauge coupling g cannot become zero in presence of gravity
  - Completeness [Polchinski '03]
    The entire charge lattice must be popultated by physical particles
  - Weak Gravity [Arkani-Hamed, Motl, Nicolis, Vafa '06], [Heidenreich, Reece, Rudelius '16-'17], [Montero, Shiu, Soler '16]
    A (sub)lattice/tower of charged particles with $g^2 q^2 \geq \mu M^2$ *should exist*
  - Distance Conjecture [Ooguri, Vafa '06]
    An "infinite" deformation of physics leads to a tower of particles with their masses exponentially suppressed

Quantum Gravity Conjectures

String-Theoretic Approach

- **Quantum Gravity Conjectures ~ Swampland Conjectures**
  - Consistency constraints Quantum Gravity (QG) is believed to impose
  - *Landscape* v.s. *Swampland* in the context of string theory

- **Conjectures in the Literature**
  - No Global Symmetry [Banks, Dixon '88]

    Gauge coupling g cannot become zero in presence of gravity → What happens if it does?
  - Completeness [Polchinski '03]

    The entire charge $\boxed{\textit{lattice}}$ *must be popultated by physical particles*
  - Weak Gravity [Arkani-Hamed, Motl, Nicolis, Vafa '06], [Heidenreich, Reece, Rudelius '16-'17], [Montero, Shiu, Soler '16]

    A $\boxed{\textit{(sub)lattice/tower}}$ *of charged particles with* $g^2q^2 \geq \mu M^2$ *should exist*
  - Distance Conjecture [Ooguri, Vafa '06]

    An "infinite" deformation of physics leads to a $\boxed{\textit{tower}}$ *of particles <u>with their masses exponentially suppressed</u>*

Quantum Gravity Conjectures

String-Theoretic Approach

- **Quantum Gravity Conjectures ~ Swampland Conjectures**
  - Consistency constraints Quantum Gravity (QG) is believed to impose
  - *Landscape* v.s. *Swampland* in the context of string theory

- **Conjectures in the Literature**
  - No Global Symmetry [Banks, Dixon '88]

    Gauge coupling g *cannot become zero in presence of gravity* → What happens if it does?
  - Completeness [Polchinski '03]

    The entire charge lattice must be popultated by physical particles

    6d: [S.-J.L., Lerche, Weigand '18-'19]

    4d: [Klaewer, S.-J.L., Weigand, Wiesner: to appear]
  - Weak Gravity [Arkani-Hamed, Motl, Nicolis, Vafa '06], [Heidenreich, Reece, Rudelius '16-'17], [Montero, Shiu, Soler '16]

    A (sub)lattice/tower of charged particles with $g^2q^2 \geq \mu M^2$ *should exist*
  - Distance Conjecture [Ooguri, Vafa '06]

    An "infinite" deformation of physics leads to a tower of particles with their masses exponentially suppressed

We will confirm these for "Open Strings"

Weak-Coupling Limit & a Shrinking Curve

6d F-theory [S.-J.L., Lerche, Weigand '18]

- ## The Key Geometric Statement

In the weak-coupling limit where $\mathrm{vol}_J(b)\,(\sim 1/g^2) \to \infty$ while $\mathrm{vol}_J(B_2)\,(\sim M_{\mathrm{Pl}}^4) = 1$ is fixed, one can find a shrinking curve **C_0** with the following properties:

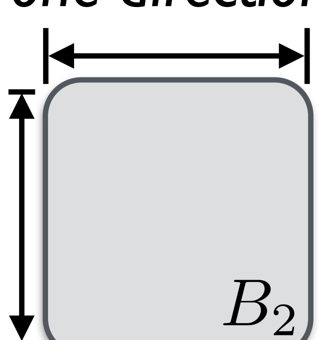
1. $\mathrm{vol}_J(C_0) \to 0$
2. $C_0 \cdot b \neq 0$
3. $C_0 \cdot C_0 = 0$
4. $C_0 \cdot \bar{K} = 2 \;\; (\, g(C_0) = 0\,)$

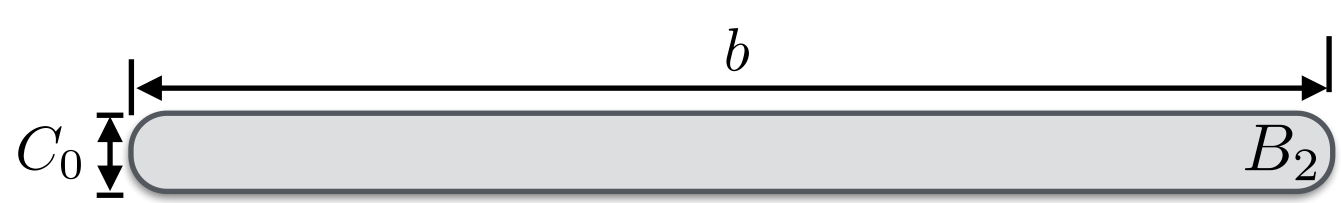
Unique,
Tensionless,
Charged,
Heterotic String

In fact, the shrinking curve class is unique (up to scaling) if we demand **$C_0 \cdot C_0 \geq 0$**

- ## An Intuition

" *If one direction is big its normal direction must be small* "





Weak-Coupling Limit & a Shrinking Curve

Existence in 6d F-theory [S.-J.L., Lerche, Weigand '18]

- **Existence Proof**

Step 1: With $\overline{\mathbf{K}(B_2)} = \langle J_0, I_\nu \rangle$, we have

$$J = tJ_0 + \sum_\nu s_\nu I_\nu$$

where $t \to \infty$ and s_ν stay finite

- For $\mathrm{vol}(b) \to \infty$, (exactly) one coefficient must go to infinity; call it t
- The volume of B2 takes the form,

$$\mathrm{vol}(B_2) = t^2 \frac{J_0 \cdot J_0}{2} + t \underbrace{n_\nu}_{J_0 \cdot I_\nu} s_\nu + \frac{s_\nu s_\mu \underbrace{n_{\nu\mu}}_{I_\nu \cdot I_\mu}}{2}$$

- Importantly, for vol(B2) = 1 with $t \to \infty$,

$$\boxed{J_0 \cdot J_0 = 0} \quad \text{and} \quad \boxed{n_\nu s_\nu \to \frac{1}{t}}$$

Step 2: J_0 is the class of a holomorphic curve C_0 with the desired properties

1. $\mathrm{vol}(C_0) \to 0$

$$\mathrm{vol}(C_0) = J \cdot C_0 = n_\nu s_\nu \to \frac{1}{t}$$

2. $C_0 \cdot b \neq 0$

$$\mathrm{vol}(b) = J \cdot b = t \underbrace{\boxed{C_0 \cdot b}}_{=2m} + \cancel{s_\nu}^{\,\mathbf{0}} I_\nu \cdot b \to \infty$$

3. $C_0 \cdot C_0 = 0$

4. $C_0 \cdot \bar{K} = 2$

$$2g(C_0) - 2 = C_0 \cdot (C_0 + K) = -J_0 \cdot \bar{K} \not\geq 0$$

$$\Rightarrow \quad g(C_0) = 0\,, \quad C_0 \cdot \bar{K} = 2$$

Weak-Coupling Limit & a Shrinking Curve

Uniqueness in 6d F-theory [S.-J.L., Lerche, Weigand '18]

- **Better be Unique** **(up to scaling)**
  - Existence has been addressed: C_0
  - Another shrinking curve C w/ $C\cdot C \geq 0$, if exists, would bring a physical problem

- **Uniqueness Proof**
  - Essentially due to the $(1, n_T)$ signature of the two-cycle intersections in B_2
  - The argument:
    - Suppose another curve C shrinks with $C\cdot C$ non-negative.
    - $C_0 = (a_0;\ a_1,\ \dots,\ a_{n_T}),\ \ C = (b_0;\ b_1,\ \dots,\ b_{n_T})$.

$$C_0\cdot C_0 = 0 \longrightarrow a_0^2 = \sum a_i^2$$

$$C_0\cdot C = J_0\cdot C = 0 \longrightarrow a_0 b_0 = \sum a_i b_i$$

- Cauchy-Schwarz gives $a_0^2\, b_0^2 = (\sum a_i b_i)^2 \leq (\sum a_i^2)(\sum b_i^2) = a_0^2 (\sum b_i^2)$
- For equality to hold, C must be proportional to C_0

Weak-Coupling Limit & a Shrinking Curve

4d F-theory [Klaewer, S.-J.L., Weigand, Wiesner: to appear]

- **The Key Geometric Statement**

In the weak-coupling limit*, there *exists* a *unique* genus-0 curve **C₀** *shrinking at a fastest rate*

with normal bundle $\mathcal{N}_{C_0/B_3} = \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_0}$.

* We restrict to those weak-coupling limits in the sense of classical geometry that keep giving a weak gauge coupling even after quantum corrections.

⇒ **Unique, Tensionless, Charged, Heterotic String**

- **What's New in 4d?**

  - Weak coupling can arise from two classes of geometric limits:

B_2 (6d)	B_3 (4d)
$J = tJ_0 + \cdots \quad , \quad t \to \infty$	$J = tJ_0 + \cdots \quad , \quad t \to \infty$
$J_0^2 = 0$	$J_0^3 = 0$ → $J_0^2 \neq 0$ ·········· *J-class A* $J_0^3 = 0$ → $J_0^2 = 0$ ·········· *J-class B*

[S.-J.L., Lerche, Weigand '19]

  - Both classes lead to the same geometric result as stated above

[Klaewer, S.-J.L., Weigand, Wiesner: to appear]

5/5. Super-Extremal Tower

Confirmation of QGCs

6d F-theory [S.-J.L., Lerche, Weigand '18]

- ## No Global Symmetries
  - The weak-coupling limit must lie at infinite distance in moduli space
  - Indeed: the limit $t \to \infty$ is located at an infinite distance $d \to \log(2t)$

- ## Distance Conjecture
  - A tower of states must become light (w/ mass supp. exponentially by d)
  - Indeed: the shrinking curve, once wrapped by D3, leads to *tensionless heterotic string*
  - Specifically:

    The tension is given as $T \propto \mathrm{vol}(C_0) \simeq \dfrac{1}{t} \simeq 2e^{-d}$

    The mass scale of the tower is determined by the tension $M^2 \propto T \propto e^{-d}$

Calculation of the Elliptic Genus

6d Example: dP1 [S.-J.L., Lerche, Weigand '18]

- **A Model over** $B_2 = dP_1$
  - Topology of the model gives the U(1) elliptic index, $m = \frac{1}{2} C_0 \cdot b = 2$
  - Mirror symmetry computation gives the genus-0 prepotential

$$\begin{aligned} \mathcal{F}^{(0)}_{C_0}(\tau, z) &\equiv \sum N^{(0)}_{C_0}(n, r)\, q^n \xi^r \\ &= -2 + \left(216 + 128\xi^{\pm 1} + 4\xi^{\pm 2}\right) q \\ &\quad + \left(121964 + 70528\xi^{\pm 1} + 9808\xi^{\pm 2} + 128\xi^{\pm 3} - 2\xi^{\pm 4}\right) q^2 \\ &\quad + \mathcal{O}(q^3)\,. \end{aligned}$$

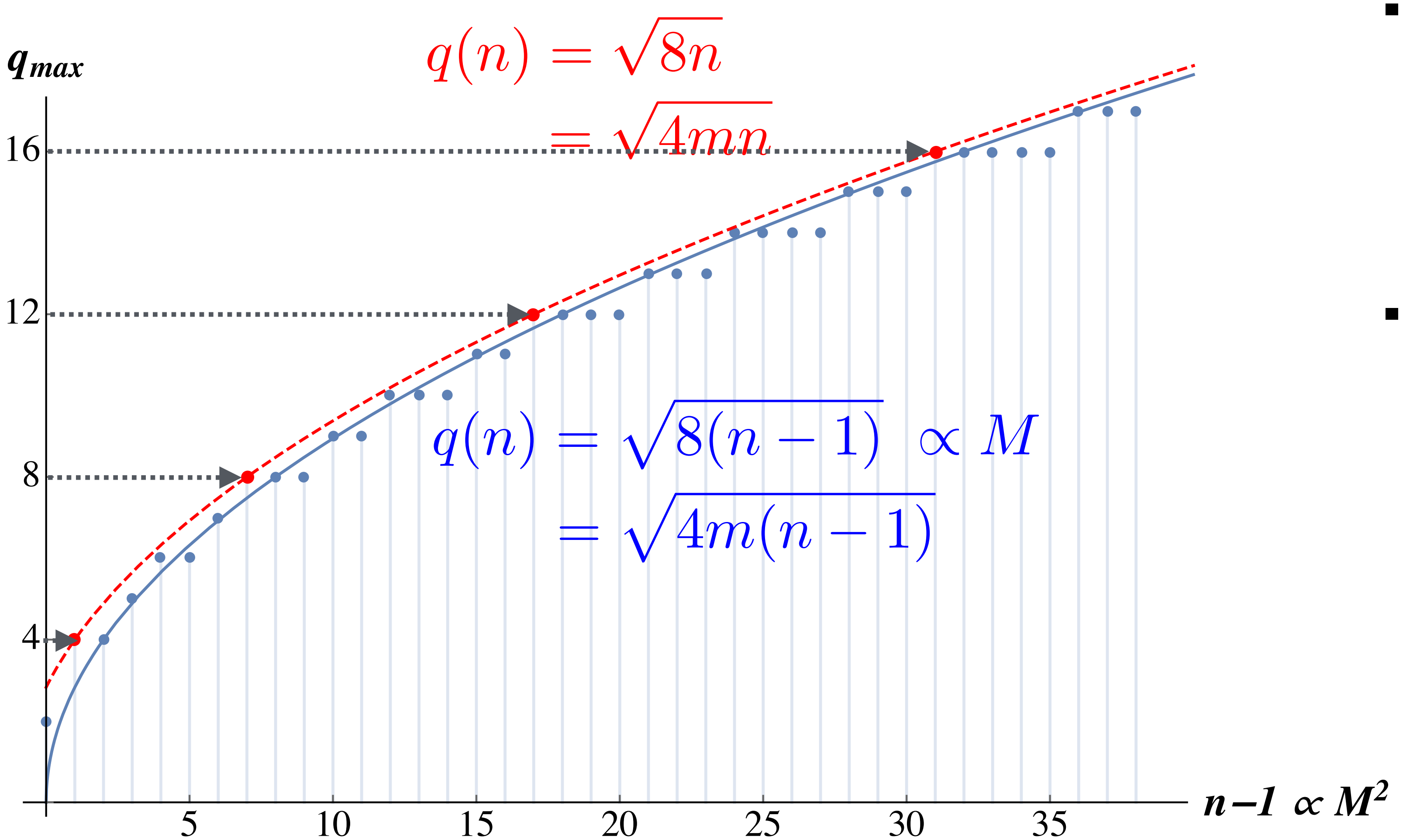
with $\xi^{\pm n} := \xi^n + \xi^{-n}$

  - Modular property fixes its analytic expression as

$$\begin{aligned} \mathcal{F}^{(0)}_{C_0}(\tau, z) &= -q\, Z_{\mathcal{K}}(\tau, z) \\ &= \frac{q}{\eta^{24}} \left(\frac{1}{54} E_4^3 \varphi_{-2,1}\varphi_{0,1} + \frac{1}{108} E_6^2 \varphi_{-2,1}\varphi_{0,1} - \frac{1}{72} E_4^2 E_6 \varphi_{-2,1}^2 - \frac{1}{72} E_4 E_6 \varphi_{0,1}^2 \right) \end{aligned}$$

Calculation of the Elliptic Genus

6d Example: dP1 [S.-J.L., Lerche, Weigand '18]



- For $q_k = 4k = 2mk$ states exist on the red curve w/ $n_k = 2k^2 = mk^2$

- The *theta expansion*: guarantees that every charge on this (sub-) lattice has states on the red curve, and in particular, strictly above the blue curve

$$\phi(\tau, z) = \sum_{l \in \mathbb{Z}/2m\mathbb{Z}} h_l(\tau) \vartheta_{m,l}(\tau, z)$$

$$\vartheta_{m,l}(\tau, z) := \sum_{r \equiv l \mod 2m} q^{\frac{r^2}{4m}} \xi^r$$

Confirmation of QGCs, Continued

6d F-theory [S.-J.L., Lerche, Weigand '18]

- **Completeness Hypothesis**
  - The entire charge lattice is populated by physical states
  - In fact, open string sector alone would not satisfy this

- **Weak Gravity Conjectures** (Sublattice, Scalar version)
  - Masses of the perturbative heterotic string excitations: $M_n^2 = 8\pi T(n-1)$
  - Consider the sublattice states with $q_k = 2mk$ and $n_k = mk^2$

sub-lattice index:
$2m = C_0 \cdot b$

Carefully keeping the numerical factors, first get

$$M_{\rm Pl}^4 = 4\pi \mathrm{vol}(B_2)\,; \qquad \frac{1}{g^2} = \frac{1}{2\pi}\mathrm{vol}(b)\,; \qquad T = 2\pi \mathrm{vol}(C_0)\,,$$

and then, observe $\mathrm{vol}(C_0)\,\mathrm{vol}(b) \to 2m\,\mathrm{vol}(B_2)$ in the weak-coupling limit; this implies that

$$\boxed{g^2 q_k^2 = \frac{M_{n_k}^2}{M_{\rm Pl}^4} + 4mg^2} > \frac{M_{n_k}^2}{M_{\rm Pl}^4} = \mu \frac{M_{n_k}^2}{M_{\rm Pl}^4} \quad \text{with} \quad \mu = 1$$

Confirmation of QGCs, Continued

6d F-theory [S.-J.L., Lerche, Weigand '18]

- Completeness Hypothesis
  - The entire charge lattice is populated by physical states
  - In fact, open string sector alone would not satisfy this

- Weak Gravity Conjectures (Sublattice, Scalar version)
  - Masses of the perturbative heterotic string excitations: $M_n^2 = 8\pi T(n-1)$
  - Consider the sublattice states with $q_k = 2mk$ and $n_k = mk^2$

sub-lattice index: $2m = C_0 \cdot b$

Carefully keeping the numerical factors, first get

$$M_{\rm Pl}^4 = 4\pi \mathrm{vol}(B_2)\,; \qquad \frac{1}{g^2} = \frac{1}{2\pi}\mathrm{vol}(b)\,; \qquad T = 2\pi \mathrm{vol}(C_0)\,,$$

and then, observe $\mathrm{vol}(C_0)\,\mathrm{vol}(b) \to 2m\,\mathrm{vol}(B_2)$ in the weak-coupling limit; this implies that

$$g^2 q_k^2 = \frac{M_{n_k}^2}{M_{\rm Pl}^4} + 4mg^2 > \frac{M_{n_k}^2}{M_{\rm Pl}^4} = \mu \frac{M_{n_k}^2}{M_{\rm Pl}^4} \quad \text{with} \quad \mu = 1$$

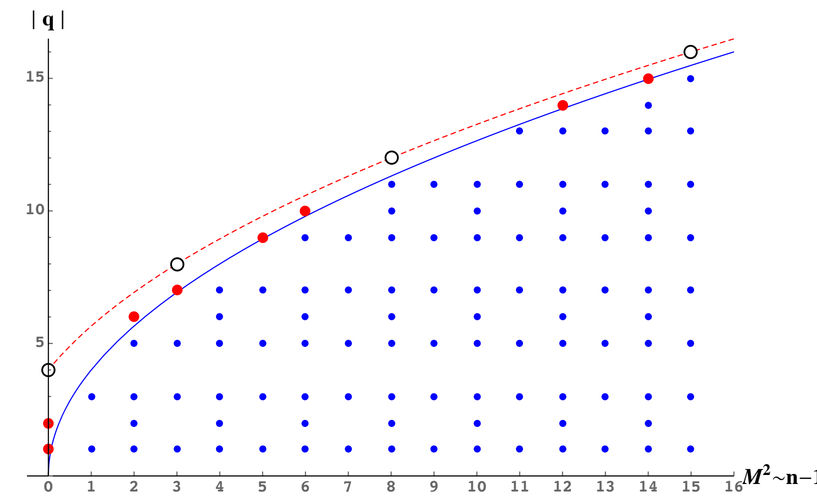
Weak Gravity Conjectures in 4 Dimensions

Special vs. General Fluxes [S.-J.L., Lerche, Weigand '19] [Klaewer, S.-J.L., Weigand, Wiesner: to appear]

- **Special (quasi-modular) Fluxes**
  - The 4d heterotic elliptic genus is modular/Jacobi

    — *theta expansion* of Jacobi forms (wt. *-1*)

    — a (shifted) sublattice of super-extremal particles!



- **General Fluxes**
  - The elliptic genus of 4d heterotic string is generically ***not*** modular/Jacobi
  - The deviation is controllable, nevertheless [S.-J.L., Lerche, Lockhart, Weigand '20]

$$\mathcal{Z}_{C_0;G}(q,\xi) = Z_{-1,m}(q,\xi) + \xi\partial_\xi Z_{-2,m}(q,\xi)$$

  - In the derivative part:

    — super-extremal terms in $Z_{-2,m}$ guaranteed by the modular/Jacobi property

    — state counting in each (n, r)-sector multiplied by r due to the derivative

    — existence of charged super-extremal sublattice guaranteed!

Weak Gravity Conjectures in 4 Dimensions

New Features in 4 Dimensions [Klaewer, S.-J.L., Weigand, Wiesner: to appear]

- **What's New in 4d?**

(a) Quantum corrections do change the physics, but the weak gravity still holds:

— $\mathrm{vol}(C_0)\,\mathrm{vol}(b) \to 2m\,\mathrm{vol}(B_3)$ before/after (leading) corrections [Grimm, Savelli, Weissenbacher '13] [Weissenbacher '20]

(b) The U(1)s are generically Stuckelberg massive:

— weak gravity still works for Stuckelberg U(1)s (cf.) [Reece '18]

— the mass is effectively zero for special (quasi-modular) fluxes [S.-J.L., Lerche, Weigand '19]

(c) U(1)s can be obtained also by breaking non-abelian algebra via flux

— a tower of states that are "super-extremal w.r.t Cartan U(1)s" guaranteed by the modular/Jacobi property (of a *Weyl-invariant* Jacobi form)

— weak gravity thus holds after the flux breaking

Conclusions

Summary

We have studied elliptic genera of N=(0,2) strings in 4 dimensions

- Solitonic strings in F-theory compactifications on a general elliptic 4-fold
- Elliptic genera obtained via GW invariants
- Behavior of elliptic genera under modular/elliptic transformations clarified:

  $$\mathcal{Z}_{C_{\mathrm{b}};G}(q,\xi) = Z_{-1,m}(q,\xi) + \xi\partial_\xi Z_{-2,m}(q,\xi)$$

  — manifestation of the quasi-Jacobi nature
  — connection b/w elliptic 4- and 3-fold invariants
  — connection b/w (-1)- and (-2)-flux invariants

- Explicit examples worked out via mirror symmetry for "GLSM 4-folds"
- Weak gravity conjectures (tower/sublattice ver.) verified
  — for general 4d N=1 EFTs of F-theory with fluxes
  — immediate consequence of the modular/elliptic behavior of the heterotic elliptic genus

Summary

We have studied elliptic genera of N=(0,2) strings in 4 dimensions

- Solitonic strings in F-theory compactifications on a general elliptic 4-fold
- Elliptic genera obtained via GW invariants
- Behavior of elliptic genera under modular/elliptic transformations clarified:

$$\mathcal{Z}_{C_{\mathrm{b}};G}(q,\xi) = Z_{-1,m}(q,\xi) + \xi\partial_\xi Z_{-2,m}(q,\xi)$$

— manifestation of the quasi-Jacobi nature
— connection b/w elliptic 4- and 3-fold invariants
— connection b/w (-1)- and (-2)-flux invariants

- Explicit examples worked out via mirror symmetry for "GLSM 4-folds"
- Weak gravity conjectures (tower/sublattice ver.) verified

— for general 4d N=1 EFTs of F-theory with fluxes
— immediate consequence of the modular/elliptic behavior of the heterotic elliptic genus

Thank You!