The logarithmic gauged linear sigma model : Theory

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Input (on GW side)

X : smooth proj. variety/DM stack e.g. $\mathbb{P}^4$

V : v.b./X e.g. $V = \mathcal{O}_{\mathbb{P}^4}(+5)$

$W \in H^0(V)$ s.t. $Y = (W=0) \subset X$

$\Updownarrow$ smooth of codim $Y/X = \mathrm{rk}\, V$

$W : V^\vee \longrightarrow \mathbb{C}$ s.t. $\underbrace{Y = (dW=0)}$

$\circlearrowleft$

$\mathbb{C}^* = \mathbb{C}^*_\omega$-action.

$\Rightarrow (\, \mathbb{P}^0 = V^\vee,\ W,\ \overbrace{\mathbb{C}^*_\omega}\,)$ R-charge.

<u>Rmk</u> (1) $\mathbb{P}^0$ needs not to be v.b.

(2) $\mathbb{P}^0$ can't be proper.

stable maps with p-fields

C : nodal curve

(no markings for simplicity)

$\Rightarrow$ dualizing l.b : $\mathring{\omega}_C = \omega_C \setminus 0_{\omega_C} \circlearrowleft \mathbb{C}^* = \mathbb{C}^*_\omega$

$$\begin{tikzcd} {[\mathbb{P}^0 \times \mathring{\omega}_C / \mathbb{C}^*_\omega] =: \mathcal{P}^\circ_C} \arrow[d] \\ C \arrow[u, bend right, "f"'] \end{tikzcd}$$

f: map with p-field.

p-field.

$$f \iff \left(C \xrightarrow{h} X \,,\ p \in H^0(f^* V^\vee \otimes \omega_C) \right)$$

f stable iff h is stable

GLSM-instanton moduli

$$\mathcal{M}^p := \{ f : C \to \mathcal{P}^\circ_C \text{ stable} \} / \sim$$

$$\cup$$

$$\mathcal{M}(X)$$

?

$$\left.\begin{array}{c} \cup \\ \mathcal{M}(Y=(dW=0)) \end{array}\right\} \; p=0$$

(1) $\mathcal{M}^p$ has $\mathbb{C}^*_\omega$-action

(2) It recovers GW of $Y=(dW=0)$:

Kiem-Li cosection localization:

$$[\mathcal{M}^p]_\sigma \in A_*(\mathcal{M}(Y))$$

rely on $(\mathbb{P}^\circ, W, \mathbb{C}^*_\omega)$

s.t. (a) $[\mathcal{M}^p]_\sigma = \pm [\mathcal{M}(Y)]^{vir}$

(b) $\iota_*[\mathcal{M}^p]_\sigma = [\mathcal{M}^p]^{vir}$ $\rightsquigarrow$ rely on $(\mathbb{P}^\circ, \mathbb{C}^*_\omega)$

Chang-Li, Kim-Oh, Chang-M. Li

C-J-Webb, R. Picciotto.

(3) $\mathcal{M}^p$ is non-proper!

We fix this :

Thm (CJRS, CJR)

There is a compactification

$$\mathcal{M}^p \subset \mathcal{U}^p$$

s.t.

(1) $\mathcal{U}^p$ is proper. DM, with $\mathbb{C}^*_\omega$-action.

(2) $\exists$ vir. cycles :

$$[\mathcal{U}^p]^{vir}, \quad \underbrace{[\mathcal{U}^p]^{red}, \quad [\mathcal{U}^p \setminus \mathcal{M}^p]^{red}}_{\substack{\text{depend on} \\ (\mathbb{P}^\circ \subset \mathbb{P},\ \mathbb{C}^*_\omega,\ W)}}$$

depend on $(\mathbb{P}^\circ \subset \mathbb{P},\ \mathbb{C}^*_\omega)$

s.t. (a) $\iota_* [\mathcal{M}^p]_\sigma = [\mathcal{U}^p]^{red}$

(b) $[\mathcal{U}^p]^{vir} = [\mathcal{U}^p]^{red} + m \cdot [\mathcal{U}^p \setminus \mathcal{M}^p]^{red}$

m: order of poles of W along $\infty = \overline{\mathbb{P}} \setminus \mathbb{P}^\circ$.

① How do we compactify $\mathcal{M}^p$?

② How to construct the reduced theory ?

For ① : use stable log maps of

Abramovich - C - Gross - Siebert

1. $$\begin{array}{ccc} \mathbb{P}^\circ & \subset & \mathbb{P} \circlearrowleft \mathbb{C}^*_\omega \\ \| & & \| \text{ as a log variety.} \\ V^\vee & \subset & (\mathbb{P}(V^\vee \oplus \mathcal{O}), \infty) \end{array}$$

2. $$\begin{tikzcd} \mathcal{P}_C := \left[\mathbb{P} \times \mathring{\omega}_C / \mathbb{C}^*_\omega \right] \arrow[d] \\ C \arrow[u, bend right, "f"'] \arrow[d, "\text{log curve}"] \\ S \end{tikzcd}$$

f : log map.

Stability ① f : representable

② $(\omega^{\log})^{1+\delta} \otimes f^* \mathcal{H}^k \otimes f^* \mathcal{O}(\infty) > 0$

for $k \gg 0$, $l \gg \delta \gg 0$.

H: ample class on X.

$\Rightarrow \quad \mathcal{M}^{\circ} \subset \overline{\mathcal{M}}^{P}$

For ②

$$
\begin{tikzcd}
\mathcal{U}^P \arrow[r, "\text{principlization of } \overline{\mathcal{M}}^P \setminus \mathcal{M}^P"] & \overline{\mathcal{M}}^P \\
\text{log R-maps with special configuration} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \text{log R-maps} \arrow[u, phantom, "\rotatebox{90}{$\in$}"]
\end{tikzcd}
$$

$\Rightarrow$ Kiem-Li cosection extends to $\mathcal{U}^P \setminus \mathcal{M}^P$ nicely : ① non-degenerate

② pole of order = pole of order of W along ∞.

$\Rightarrow \quad [\mathcal{U}^P]^{\mathrm{red}}$.