# CY components of GLSMs

(Work with Ballard-Katzarkov & Kelly)

# Thm (Orlov)

- $Z(f) \subseteq \mathbb{P}^n$ smooth proj. hypersurface
- $f$ is homo, of deg $d$ in $n+1$ variables

① If $d = n+1$ (CY cond) then $D(Z(f)) \cong D(\mathbb{C}^{n+1}, \mathbb{C}^\times, f)$

Matrix Factorizations

② If $d > n+1$

$$D(Z(f)) \hookrightarrow D(\mathbb{C}^{n+1}, \mathbb{C}^\times, f)$$

③ If $d < n+1$ $D(\mathbb{C}^{n+1}, \mathbb{C}^\times, f) \hookrightarrow D(Z(f))$

$D(\mathbb{C}^n, \mathbb{C}^\times, f)$

means the homotopy category of graded matrix factorizations:

Ob $\quad E_0 \xrightarrow{\alpha} E_1 \xrightarrow{\beta} E_0(d)$

$E_i$, free $\mathbb{C}[x_0, \dots, x_n]$-modules

$\alpha\beta = \beta\alpha = f.$

Mor $\quad$ Chain maps up to homotopy.

**Fact** If

$f$ is an isolated singularity (@ 0) there is a Serre functor

$$S = (d-n-1)[n-2]$$

i.e.

$$\mathrm{Hom}(E_\bullet, F_\bullet) = \mathrm{Hom}(F_\bullet, E_\bullet(d-n-1)[n-2])^*$$

internal grading shift of a graded module

homological grading shift of a complex

$$E_0 = E_0 \xrightarrow{\alpha} E_1 \xrightarrow{\beta} E_0(d)$$

$$E_0[1] = E_1 \to E_0(d) \to E_1(d)$$

$$\therefore [2] = (d)$$

Recall

$$S = (d-n-1)[n-2]$$

$$\therefore S^d = [2d-2n-2+dn-2d]$$

i.e. some power of the Serre fun. is a homological shift i.e. $D(\mathbb{C}^n, \mathbb{C}^\times, f)$ is fractional CY

**Def'n** $\mathcal{C}$ cat. w/ Serre functor

If $S = [n]$ this is called a CY cat. of dim $n$.

If $S^d = [t]$ this is Frac. CY of dim $t/d$.

Consequence of Orlov's Thm: | choices of stability for this GLSM.

① $D(Z(f)) \hookrightarrow FCY$

② OR $\hookleftarrow$

③ OR $=$

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$$\begin{array}{ccccc} x_0 & & x_n & p & \\ \left[\, 1 \right. & \cdots & 1 & \left. -d \,\right] & \mathbb{C}^{\times} \\ \left[\, 0 \right. & \cdots & 0 & \left. 1 \,\right] & \mathbb{C}^{\times 2} \end{array}$$

R-charge

$\mathbb{C}^\times$ weights $\circlearrowright \mathbb{C}^{n+2}$

$W = p\, f(x_0, \ldots, x_n)$.

Let $\mathbb{C}^\times \curvearrowright X$

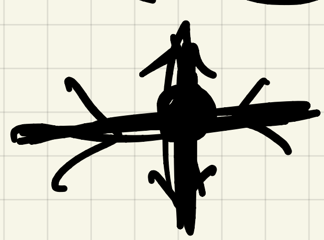
fixed locus $X^{\mathbb{C}^\times} = \{ x \in X \mid \tau \cdot x = x \ \forall \tau \in \mathbb{C}^\times \}$

$S_+ := \{ x \in X \mid \lim_{\tau \to 0} \tau \cdot x \text{ exists} \}$

$S_- := \{ x \in X \mid \lim_{\tau \to \infty} \tau \cdot x \text{ exists} \}$

<u>Example</u> $\mathbb{C}^\times \curvearrowright \mathbb{C}^2$ $[1 \ -1]$

$X^{\mathbb{C}^\times} = (0,0)$



$S_+ = (x,0)$ x axis (y=0

$S_- = (0,y)$ y axis (x=0)

**Ex** $\mathbb{C}^\times \curvearrowright \mathbb{C}^{n+2}$

$1 \ldots\ldots 1 \; -d$

$S_+ = 0 \times \mathbb{C}$

$S_- = \mathbb{C}^{n+1} \times 0$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Assume everything is smooth & connected.

You get: $N_{X^{\mathbb{C}^\times}/S_\pm}$ $\quad p \in X^{\mathbb{C}^\times}$

$\left(N_{X^{\mathbb{C}^\times}}|_{S_+} \oplus N_{X^{\mathbb{C}^\times}}|_{S_-}\right)_p$ ← graded vector space

Let $\mu = \sum$ weights $\left(\mathcal{N}|_{X^{\not\subset X}/S^+} \oplus \mathcal{N}|_{X^{\not\subset X}/S^-}\right)\Big|_p$

<u>Thm</u> (Ballard–F–Katzarkov, Halpern-Leistner)

① If $\mu = 0$

$$D\left(\frac{X|S_+}{\mathbb{C}^\times}\right) \cong D\left(\frac{X|S_-}{\mathbb{C}^\times}\right)$$

② If $\mu < 0$

$$D\left(\frac{X|S_+}{\mathbb{C}^\times}\right) \hookrightarrow D\left(\frac{X|S_-}{\mathbb{C}^\times}\right)$$

③ If $\mu > 0$ $\quad D\left(\frac{X|S_-}{\mathbb{C}^\times}\right) \hookrightarrow D\left(\frac{X|S^+}{\mathbb{C}^\times}\right)$

# GLSM version

- $\Gamma \circlearrowright \mathbb{C}^n$
- $\Gamma \xrightarrow{\chi} \mathbb{C}^\times$
- $w : \mathbb{C}^n \longrightarrow \mathbb{C}$

  $w(g \cdot x) = \chi(g)\, w(x)$

  $\forall\; g \in \Gamma$
- $\theta : \ker \chi \to \mathbb{C}^\times$

  (determines stability)

Choose $\mathbb{C}^\times \subseteq \ker \chi$

$\mathbb{C}^\times \curvearrowright \mathbb{C}^n$

Let $\mu = \sum$ weights of $\mathbb{C}^\times \curvearrowright \mathbb{C}^n$

① If $\mu = 0$

$$D(\mathbb{C}^n \setminus S_+, \Gamma, w) \cong D(\mathbb{C}^n \setminus S_-, \Gamma, w)$$

② If $\mu < 0$

$$\longleftarrow\!\!\!\longrightarrow$$

③ If $\mu > 0$

$$\hookleftarrow$$

**Warning** $D(\mathbb{C}^n \setminus S_+, \Gamma, w)$

is not quite the homotopy category of matrix factions

you need to take injective replacements

$$\mathcal{E}_0 \xrightarrow{\alpha} \mathcal{E}_1 \xrightarrow{\beta} \mathcal{E}_0(\chi)$$

injective sheaves

$D(X, \mathbb{C}^*, 0) \cong D(X)$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow 2 \\ \leftarrow r\ \text{charge} \end{matrix}$$

- $\Gamma = (\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^6$
- $\chi \sim \Pi_{\mathbb{Z}}$
- $W = p \sum_{i=0}^{4} x_i^5$
- $\theta_+ = Id : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$
- $\theta_- = inv : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$

$$\left.\begin{aligned} S_+ &= \mathbb{C}^{n+1} \times 0 \\ S_- &= 0 \times \mathbb{C} \end{aligned}\right\} \begin{matrix} \theta_+ \\ \theta_- \end{matrix} \text{ unstable loci}$$

$$\mathcal{M} \simeq 1+1+1+1+1-5 \simeq 0$$

$$D(\mathbb{C}^6, \Gamma, \theta_+, \omega) \overset{BFK}{=\!=} D(\mathbb{C}^6, \Gamma, \theta_-, \omega$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$D(\mathrm{tot}\,\mathcal{O}_{\mathbb{P}^4}(-5), \mathbb{C}^\times, \langle -, \textstyle\sum x_i^5 \rangle) \qquad D(\mathbb{C}^5, \mathbb{C}^\times, \textstyle\sum x_i^5$$

$$\| \text{ Isik-Shipman} \qquad\qquad \swarrow\!\!\nearrow \text{ Orlov}$$

$$D(Z(\textstyle\sum x_i^5))$$

# Abelian GLSMs

Assume

$$T := \ker \chi \subseteq \Gamma \subseteq Gl_n(\mathbb{C})$$

$$\subseteq (\mathbb{C}^\times)^n$$

ie $T$ is diagonal.

We get $(\mathbb{C}^\times)^n/T$

$$\begin{array}{c} \hspace{7em} \| \\ 0 \to T \to (\mathbb{C}^\times)^n \to C \to 0 \end{array}$$

Apply $\mathrm{Hom}(-, \mathbb{C}^\times)$

$$0 \to \hat{C} \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} \hat{T} \to 0$$

$\hat{C} = \mathbb{Z}^m$

$Q$ ← charge matrix.

rows of $A$ are vectors in $\mathbb{R}^m$

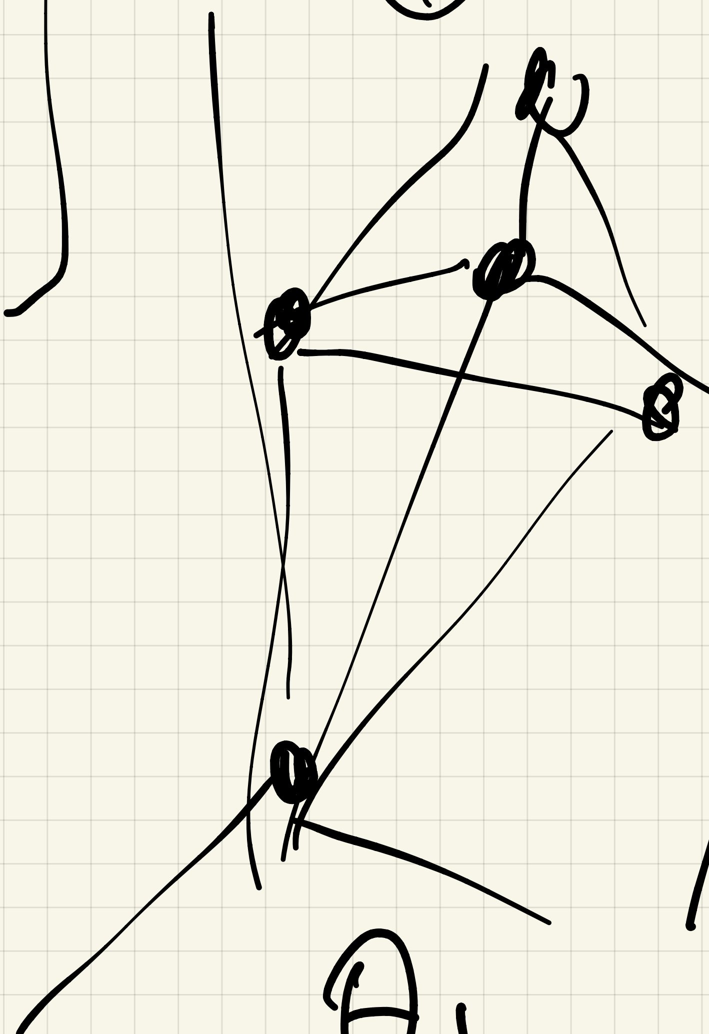
**Ex**

$$0 \to \mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^4 \xrightarrow{Q} \mathbb{Z} \to 0$$

$$Q = \begin{bmatrix} 1 & 1 & 1 & -3 \end{bmatrix}$$

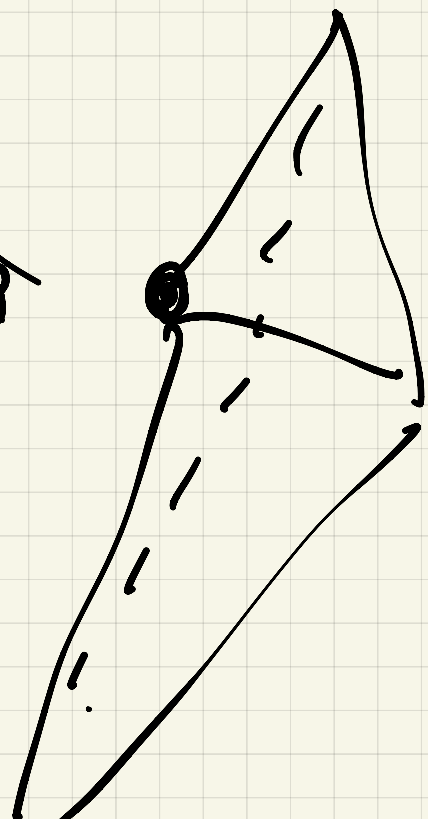
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

①



$\Theta_+$

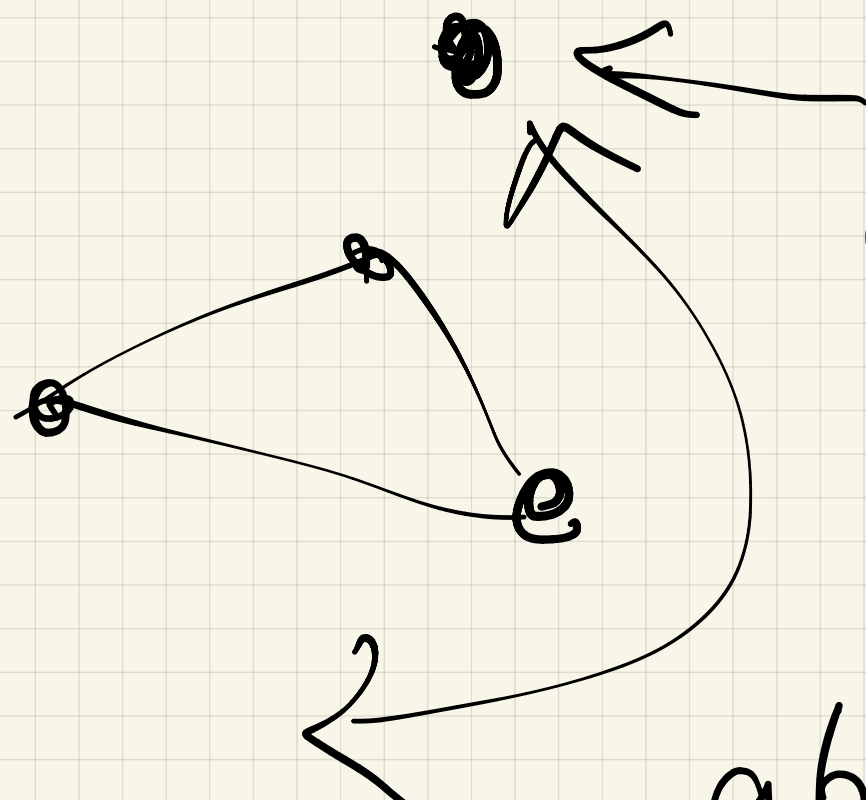
②



$\Theta_-$

more generally,

1 1 1 -d



height depends on d.

above or below this plane.

<u>**Thm**</u> (F–Kelly)

$\Gamma \circlearrowright X$ $\quad \chi : \Gamma \to \mathbb{C}^\times$

↑ linearly reductive ↑ smooth

$w \in H^0(\mathcal{O}_X(\chi))^\Gamma$

<u>Assume</u>

- $\partial w$ is proper
- $[X/\ker\chi]$ is DM
- $\partial w \subseteq Z(w)$
- $[X/\ker\chi]$ has torsion canonical bundle

Then $D(X, \Gamma, w)$ is Frac. CY.

Thm (F-Kelly

Orlov's Thm for GLSMs)

- Let $C = \mathrm{Cone}(r_1^A, \ldots, r_n^A)$

$$0 \to \mathbb{Z}^m \xrightarrow{A} \mathbb{Z}^n \xrightarrow{Q} \hat{T} \to 0$$

① Assume the ray generators of $C \cap \mathbb{Z}^m$ lie in a $\mathbb{Q}$-plane. Then the GLSM admits a Frac CY phase

② — // — $\mathbb{Z}$-plane

Then — // — CY phase.

(3) Lets call this (hyper) plane $P$.

& assume all $r_i^A$ lie on or above $P$.

Then $\forall$ generic stab. cond. $\theta$

$D(\mathbb{C}^n, \Gamma, w, \theta_{FCY}) \longleftrightarrow D(\mathbb{C}^n, \Gamma, w, \theta)$

(4) Assume all $r_i^A$ lie on or below $P$

Then —"— $\hookleftarrow$ —"— .

# Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & -3 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

+ R-charge

## rows

$$e_1, \dots, e_5, 3e_6 - e_1 - \dots - e_5,$$

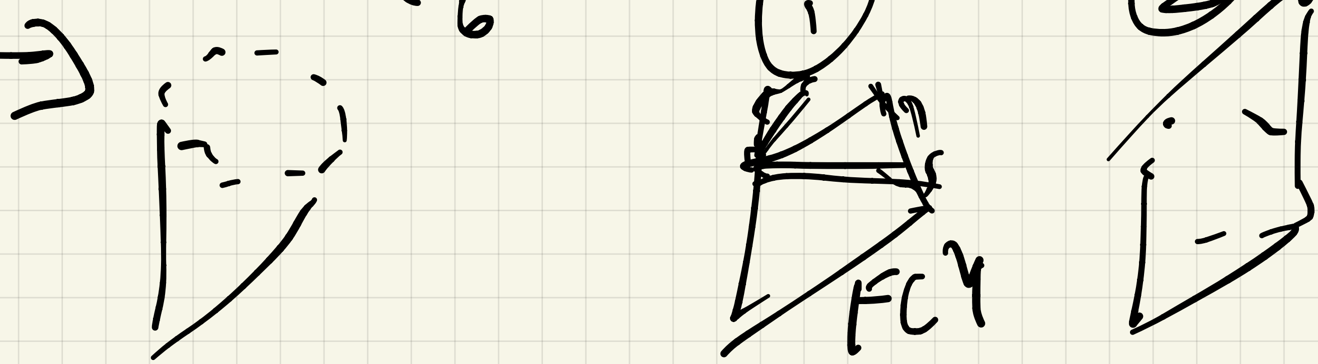
$$e_6$$

height 1

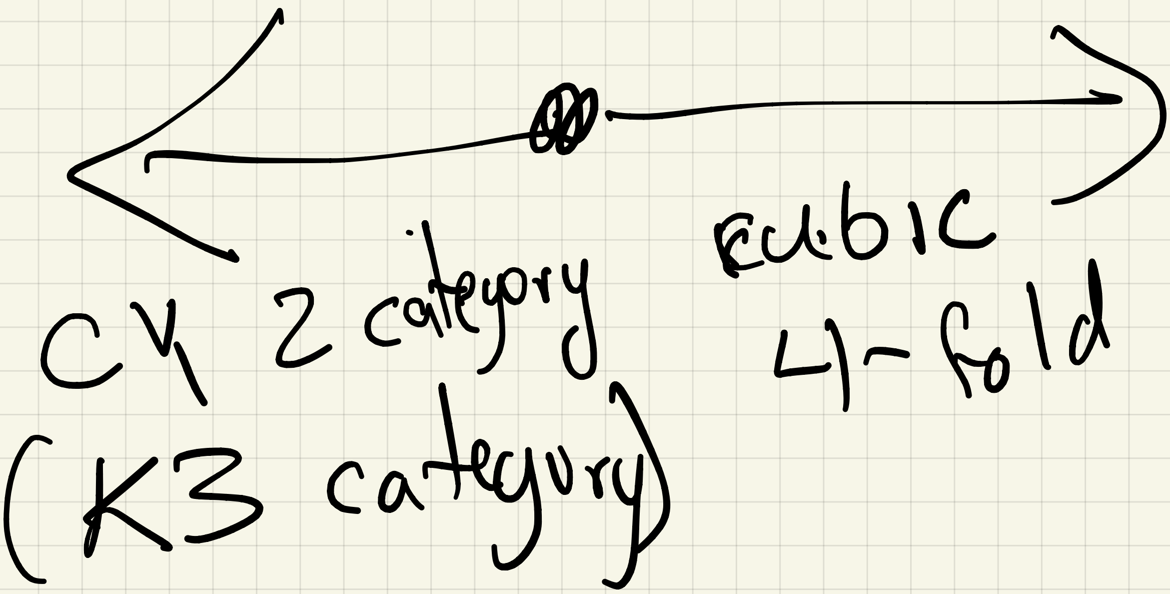
height 2

$$e_1^* + \dots + e_5^* + 2e_6^*$$

$$\cdot\ e_6$$



$[1\ 1\ 1\ 1\ 1\ 1\ -3]$




$$x_0 f(x_1, \ldots, x_5) + g(x_1, \ldots, x_5)$$

generically has 1 singular point.

$A$ • $D(X_3)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 \end{bmatrix}$$

open immersion of GLSM!

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

2| phases:

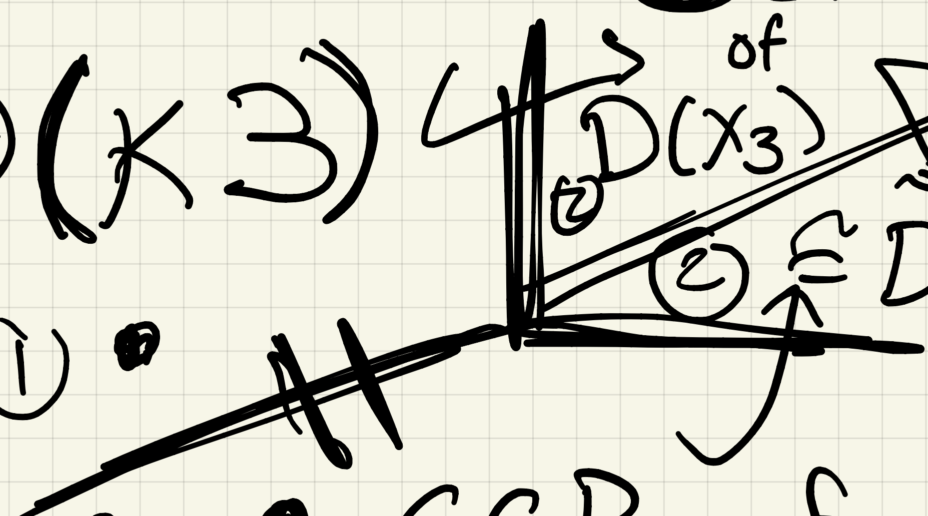
4 dim'l

CCR of $D(X_3)$

$D(K3) \hookrightarrow$ ② $D(X_3)$

② $\subset D(\tilde{X}_3)$ (tot $\mathcal{O}(-3H+E)$, $Bl_p\mathbb{P}^5$)

not crepant.



2 dim'l

① • CCR of $A$.

2 dim'l CY.