Quantum Sheaf Cohomology and Duality of Flag Manifolds

Jirui Guo

Yau Mathematical Sciences Center, Tsinghua University

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- Ring structures have been solved for toric varieties [McOrist, Melnikov, 07'] [Donagi, Guffin, Katz, Sharpe, 11'], Grassmannians [JG, Lu, Sharpe, 15'] and flag manifolds [JG 18'].

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- OPE $\rightarrow$ Quantum Sheaf Cohomology

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- An operator $\mathcal{O}_R$ is zero in the quantum sheaf cohomology if and only if the $A/2$ correlation function $\langle \mathcal{O}_R \mathcal{O} \rangle = 0$ for any operator $\mathcal{O}$.

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- In the classical limit, localization on Coulomb branch [Closset, Gu, Jia, Sharpe, 15'] implies

$$\mathrm{Res}_{(0)} \frac{\Delta^2 \mathcal{O}_R \mathcal{O}}{\prod_\gamma \prod_{\rho_\gamma \in R_\gamma} \left(\det(M_{(\gamma,\rho_\gamma)})\right)} d\sigma_1 \wedge \cdots \wedge d\sigma_{\mathrm{rk}(G)} = 0$$

$\Delta \equiv \prod_{\alpha>0} \alpha(\sigma)$. α : positive roots of the gauge group.

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- $\Delta^2 \cdot \mathcal{O}_R$ must lie in the ideal generated by $\det(M_{(\gamma,\rho_\gamma)})$ with γ and ρ_γ running over all the representations and weights of the matter multiplets.
- If on the (2,2) locus the quantum cohomology has a representation $\mathcal{A}/(I+R)$, then there is a set of generators h_r of R such that for each r, $\Delta^2 h_r$ is a function of $\det(M_{(\gamma,\rho_\gamma)})$,i.e.

$$\Delta^2 h_r = P_r(\det(M_{(\gamma,\rho_\gamma)})).$$

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- Quantum relations: $J^a = 0 \Rightarrow \tilde{h}_r(q) = 0$.

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- With deformation given by $\overline{D}_+ \Lambda^i_\alpha = \Sigma^\beta_\alpha \Phi^i_\beta + A^i_j(\mathrm{Tr}\Sigma)\Phi^j_\alpha$, the generators of R:
  $y_{N-k+r} \to R_{N-k+r} = \sum_{i=0}^{\min\{N,N-k+r\}} I_i(y_1 A) y_{N-k+r-i}$.
  (I_i: i-th characteristic polynomial)
- quantum: $R_{N-k+r} + q y_{r-k} = 0$.

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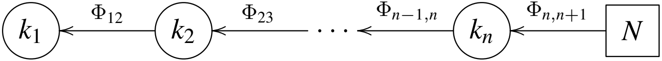

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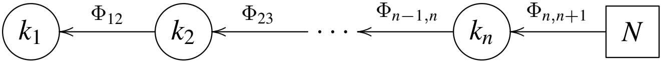

- (0,2) deformation:

$$\overline{D}_+ \Lambda_{s,s+1} = \Phi_{s,s+1}\Sigma^{(s)} - \Sigma^{(s+1)}\Phi_{s,s+1} + \sum_{t=1}^{n} u_t^s (\mathrm{Tr}\Sigma^{(t)})\Phi_{s,s+1},$$

$$s = 1, \cdots, n-1$$

$$\overline{D}_+ \Lambda^i_{n,n+1} = \Phi_{n,n+1}\Sigma^{(n)} + \sum_{t=1}^{n} (\mathrm{Tr}\Sigma^{(t)}) A_{tj}^{\ i} \Phi^j_{n,n+1}, \ i,j = 1, \cdots, N.$$

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- It can be shown that QSC reduces to QC as $u, A \to 0$.

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- Answer: $({u'}^s_t, {A'}^i_{tj})$ can be solved by QSC. (The rings on the two sides are isomorphic.)

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- Classical sheaf cohomology:

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$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \frac{1}{m} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -1 - \mathrm{Tr}(D) & \mathrm{Tr}(B) \\ \mathrm{Tr}(C) & -1 - \mathrm{Tr}(A) \end{pmatrix},$$

where $m = (1 + \mathrm{Tr}(A))(1 + \mathrm{Tr}(D)) - \mathrm{Tr}(B)\mathrm{Tr}(C)$.

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- Then the deformations are related by

$$
\begin{aligned}
B_{st} &= A_t + \sum_{a=s}^{n-1} u_t^a I, \; s \leq n-1, \\
B_{nt} &= A_t.
\end{aligned}
$$

Dual deformation of flag manifolds

- Using the results on product of Grassmannians (with dual embedding $F(N-k_n, N-k_{n-1}, \cdots, N-k_1, N) \hookrightarrow G(N-k_n, N) \times \cdots \times G(N-k_1, N)$), one can show the dual deformation on $F(N-k_n, N-k_{n-1}, \cdots, N-k_1, N)$ is given by $X' = XT^{-1}$, where $T_{ij} = \delta_{ij} + \mathrm{Tr}\left(A_j + \sum_{t=i}^{n-1} u_j^t I\right)$,

$$X_{1t} = -A_t - \sum_{l=1}^{n-1} u_t^l I,$$
$$X_{st} = u_t^{s-1},\ s = 2, \cdots, n,$$

and

$$X'_{1t} = A'_{n+1-t},$$
$$X'_{st} = u'^{n-s+1}_{n-t+1},\ s = 2, \cdots, n.$$

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