Sphere Partition Function of Calabi-Yau GLSMs

[Erkinger,Knapp,20]

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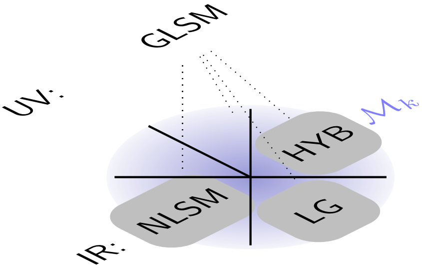
Motivation

Kähler Moduli Space

- Study **stringy Kähler moduli space** $\mathcal{M}_K$ of type II **Calabi-Yau (CY) compactifications**
- Many structures known in **geometric region**
- **GLSM** and **supersymmetric localisation**: access **other regions** [Witten,93]



Motivation

Similar Structures

- Should one expect **similar structures** throughout $\mathcal{M}_k$?
  - **worldsheet CFT** independent of compactification
  - **common UV origin**
- Structures away from geometric phase:
  - tt^* geometry [Cecotti,Vafa,91]
  - FJRW [Fan,Jarvis,Ruan,07]
  - Givental's mirror symmetry approach [Givental,97]
- We looked at the **GLSM sphere partition function** Z_{S^2} [Benini,Cremonesi,12; Doroud+,12].
  - gives **Kähler potential** K of $\mathcal{M}_K$ [Gomis,Lee,12; Gerchkovitz,Gomis,Komargodski,14; Gomis+,15]:

$$Z_{S^2} = e^{-K}.$$

GLSM Setting

- We studied **abelian GLSMs in 2 dimensions**, gauge group $U(1)^k$.
- Gauge charges of the chiral fields sum to zero (**Calabi-Yau condition**).
- k FI-θ parameters:

$$\mathsf{t}_i = 2\pi\zeta_i - i\theta_i .$$

- t_i coordinates on $\mathcal{M}_K$
- Evaluation of the **sphere partition function** Z_{S^2} is **phase dependent**.

GLSM Setting

- Studied Z_{S^2} in
  - **Geometric** phases
  - **Landau-Ginzburg** phases
  - **Hybrid** phases
- **14 one parameter** models:
  - Geometric phase:
    - $\mathbb{P}^{5+k-n-j}_{1^{5+k},\alpha^n,\beta^j}[d_1, d_2^k]$
  - **Pseudo-hybrid** phases
- **2 two parameter** models:
  - Geometric phase:
    - $\mathbb{P}_{11222}[8]$
    - $\mathbb{P}_{11169}[18]$

GLSM Setting

One Parameter List

model-data					IR-description	
label	α^n	β^j	d_1	d_2^k	$\zeta \gg 0$	$\zeta \ll 0$
			F-type			
F1	-	-	5	-	$\mathbb{P}_{1^5}[5]$	LG orbifold
F2	-	2	6	-	$\mathbb{P}_{1^4,2}[6]$	LG orbifold
F3	-	4	8	-	$\mathbb{P}_{1^4,4}[8]$	LG orbifold
F4	2	5	10	-	$\mathbb{P}_{1^3,2,5}[10]$	LG orbifold
F5	-	2	4	3	$\mathbb{P}_{1^5,2}[4,3]$	Pseudo-Hybrid
F6	2^2	3	6	4	$\mathbb{P}_{1^3,2^2,3}[6,4]$	Pseudo-Hybrid
F7	4	6	12	2	$\mathbb{P}_{1^4,4,6}[12,2]$	Pseudo-Hybrid
			C-type			
C1	-	-	4	2	$\mathbb{P}_{1^6}[4,2]$	Pseudo-Hybrid
C2	-	3	6	2	$\mathbb{P}_{1^5,3}[6,2]$	Pseudo-Hybrid
C3	-	-	3	2^2	$\mathbb{P}_{1^7}[3,2,2]$	Pseudo-Hybrid
			K-type			
K1	-	-	3	3	$\mathbb{P}_{1^6}[3,3]$	Hybrid
K2	-	2^2	4	4	$\mathbb{P}_{1^4,2^2}[4,4]$	Hybrid
K3	2^2	3^2	6	6	$\mathbb{P}_{1^2,2^2,3^2}[6,6]$	Hybrid
			M-type			
M1	-	-	2	2^3	$\mathbb{P}_{1^8}[2,2,2,2]$	Non-linear σ

Section 2

Proposal

General form of Z_{S^2}

- Given a Calabi-Yau GLSM the **sphere partition function** in a phase that is a **Landau-Ginzburg orbifold** with orbifold group G **fibered over a base manifold** B can always be written in the form:

$$Z_{S^2}(\mathsf{t},\bar{\mathsf{t}}) = \left\langle \bar{I}, I \right\rangle .$$

- **State space** associated to the phase: $\mathcal{H}$
- $\langle \cdot, \cdot \rangle$ non-degenerate **pairing** on $\mathcal{H}$
- **Gamma-class, Givental's I-function**
- Complex conjugation operation
  - Our results suggest:

$$\left\langle \overline{I} \right| = (-1)^{\mathrm{Gr}} \frac{\hat{\Gamma}}{\hat{\Gamma}^*} I(\bar{\mathsf{t}})$$

  - see [Iritani,07; Katzarkov,Kontsevich,Pantev,08; Halverson+,13]

General form of Z_{S^2}

- Given a Calabi-Yau GLSM the **sphere partition function** in a phase that is a **Landau-Ginzburg orbifold** with orbifold group G **fibered over a base manifold** B can always be written in the form:

$$Z_{S^2}(\mathsf{t},\bar{\mathsf{t}}) = C\sum_{\delta\in G}\int_B (-1)^{\mathrm{Gr}}\frac{\widehat{\Gamma}_\delta(H)}{\widehat{\Gamma}^*_\delta(H)} I_\delta(\bar{\mathsf{t}},H) I_\delta(\mathsf{t},H).$$

normalisation

twisted sector summation

integration over base manifold B
H generators of $H^2(B)$

Gr eigenvalue grading operator

$\hat{\Gamma}$ operator eigenvalue

Givental's I function

Section 3

Evidence

One Parameter Models

Geometric Phase

- B: $\mathbb{P}^{5+k-n-j}_{1^{5+k},\alpha^n,\beta^j}[d_1, d_2^k]$
- **State space**: $H^2(B)$, ambient part
- **Pairing**: Mukai-Pairing [Căldăraru,03; Halverson+,13]
  - Grading: $\alpha \in H^{2k}(B) \to (-1)^k \alpha$
- **Gamma-class:** Gamma class of B

$$Z_{S^2} = \frac{(2\pi i)^3}{\alpha^n \beta^j} \int_B \frac{\widehat{\Gamma}_B(H)}{\widehat{\Gamma}^*_B(H)} I(\bar{\mathsf{t}}, H) I(\mathsf{t}, H)$$

evaluated by using intersection ring on B

[Givental,97]

[Cox,Katz,99]

- I function observed in [Bonelli+,13; Ueda,Yoshida,16; Kim+,16; Gerhardus,Jockers,Ninad,18; Goto,Okuda,18; Honma,Manabe,18]

One Parameter Models

Geometric Phase

$$\widehat{\Gamma}_X(H) = \frac{\Gamma\left(1-\frac{H}{2\pi i}\right)^{5-n-j+k}\Gamma\left(1-\alpha\frac{H}{2\pi i}\right)^{n}\Gamma\left(1-\beta\frac{H}{2\pi i}\right)^{j}}{\Gamma\left(1-d_1\frac{H}{2\pi i}\right)\Gamma\left(1-d_2\frac{H}{2\pi i}\right)^{k}},$$

$$I(\mathsf{t},H) = \frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^{5-n-j+k}\Gamma\left(1+\alpha\frac{H}{2\pi i}\right)^{n}\Gamma\left(1+\beta\frac{H}{2\pi i}\right)^{j}}{\Gamma\left(1+d_1\frac{H}{2\pi i}\right)\Gamma\left(1+d_2\frac{H}{2\pi i}\right)^{k}}$$
$$\cdot\sum_{a=0}^{\infty}(-1)^{a(5+k-n-j+\alpha n+j\beta)}e^{-\mathsf{t}(\frac{H}{2\pi i}+a+q)}$$
$$\cdot\frac{\Gamma\left(1+ad_1+d_1\frac{H}{2\pi i}\right)\Gamma\left(1+ad_2+d_2\frac{H}{2\pi i}\right)^{k}}{\Gamma\left(1+a+\frac{H}{2\pi i}\right)^{5+k-n-j}\Gamma\left(1+a\alpha+\alpha\frac{H}{2\pi i}\right)^{n}\Gamma\left(1+a\beta+\beta\frac{H}{2\pi i}\right)^{j}}.$$

One Parameter Models

Landau-Ginzburg Phase

- B is a point.
- **State space:** FJRW-theory, narrow sector (e.g [Chiodo,Iritani,Ruan,12])
- **Pairing**: FJRW-theory
- **Gamma-class**: FJRW-theory (also [Iritani,07]).
- Hemisphere partition function [Knapp,Romo,Scheidegger,20]

$$Z_{S^2} = \frac{1}{d_1} \sum_{\delta \in narrow} (-1)^{\mathrm{Gr}} \frac{\widehat{\Gamma}_\delta(0)}{\widehat{\Gamma}^*_\delta(0)} I_\delta(\bar{\mathrm{t}}, 0) I_\delta(\mathrm{t}, 0)$$

remnant of FJRW pairing

Eigenvalue of $(-1)^{\mathrm{Gr}}$ on sector δ

Eigenvalue of $\hat{\Gamma}$ on sector δ

Givental's I function on sector δ

One Parameter Models

Landau-Ginzburg Phase

$$\widehat{\Gamma}_\delta(0) = \Gamma\left(\left\langle \frac{\delta}{d_1} \right\rangle\right)^3 \Gamma\left(\left\langle \alpha \frac{\delta}{d_1} \right\rangle\right) \Gamma\left(\left\langle \beta \frac{\delta}{d_1} \right\rangle\right),$$

$$I_\delta(\mathsf{t}, 0) = \sum_{a=0}^{\infty} \frac{e^{\mathsf{t}(a + \frac{\delta}{d_1} - q)} (-1)^{a(3+\alpha+\beta)}}{\Gamma\left(\left\langle \frac{\delta}{d_1} \right\rangle\right)^3 \Gamma\left(\left\langle \alpha \frac{\delta}{d_1} \right\rangle\right) \Gamma\left(\left\langle \beta \frac{\delta}{d_1} \right\rangle\right)} \cdot \frac{\Gamma\left(a + \frac{\delta}{d_1}\right)^3 \Gamma\left(a\alpha + \frac{\alpha}{d_1}\delta\right) \Gamma\left(a\beta + \frac{\beta}{d_1}\delta\right)}{\Gamma\left(\delta + a d_1\right)}.$$

One Parameter Models

K-Type Hybrids

- B: $\mathbb{P}^1$
- **State Space:** FJRW-theory [Clader,13; Chiodo,Nagel,18]
- **Pairing:** Z_{S^2} not conclusive
- **Gamma-class:** [Zhao,19; Iritani,07]

$$Z_{S^2} = \underbrace{\frac{2\pi i}{d_1} \sum_{\delta \in Narrow} \int_{\mathbb{P}^1}}_{\text{mixture of pairing in LG and Geometric phase}} \underbrace{(-1)^{\mathrm{Gr}} \frac{\Gamma_\delta(H)}{\Gamma^*_\delta(H)} I_\delta(\bar{\mathrm{t}}, H) I_\delta(\mathrm{t}, H)}_{\text{Similar to LG phase, but now dependent on } H}$$

mixture of pairing in LG and Geometric phase

Similar to LG phase, but now dependent on H

One Parameter Models

K-Type Hybrids

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$$\Gamma_\delta(H) = \Gamma\left(1-\frac{H}{2\pi i}\right)^2 \Gamma\left(\frac{H}{2\pi i d_1}+\left\langle\frac{\delta}{d_1}\right\rangle\right)^{6-n-j} \cdot \Gamma\left(\alpha\frac{H}{2\pi i d_1}+\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^{n} \Gamma\left(\beta\frac{H}{2\pi i d_1}+\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^{j},$$

$$\mathcal{I}_\delta(\mathsf{t},H) = \frac{\Gamma\left(1+\frac{H}{2\pi i}\right)^2}{\Gamma\left(\frac{H}{2\pi i d_1}+\left\langle\frac{\delta}{d_1}\right\rangle\right)^{6-n-j}\Gamma\left(\alpha\frac{H}{2\pi i d_1}+\left\langle\alpha\frac{\delta}{d_1}\right\rangle\right)^{n}\Gamma\left(\beta\frac{H}{2\pi i d_1}+\left\langle\beta\frac{\delta}{d_1}\right\rangle\right)^{j}} \cdot \sum_{a=0}^{\infty} e^{\mathsf{t}\left(\frac{H}{2\pi i d_1}+a+\frac{\delta}{d_1}-q\right)}(-1)^{a(6-n-j+\alpha n+j\beta)} \cdot \frac{\Gamma\left(a+\frac{H}{2\pi i d_1}+\frac{\delta}{d_1}\right)^{6-n-j}\Gamma\left(a\alpha+\alpha\frac{H}{2\pi i d_1}+\frac{\alpha}{d_1}\delta\right)^{n}\Gamma\left(a\beta+\beta\frac{H}{2\pi i d_1}+\frac{\beta}{d_1}\delta\right)^{j}}{\Gamma\left(\delta+ad_1+\frac{H}{2\pi i}\right)^2}.$$

One Parameter Models

M-Type Hybrids

- B: $\mathbb{P}^3$
- Similarities to geometric phase
- Described as hybrid [Clader,13]
- State space, pairing, gamma-class: see K-type

$$Z_{S^2} = \frac{(2\pi i)^3}{2} \int_{\mathbb{P}^3} (-1)^{\mathrm{Gr}} \frac{\Gamma_1(H)}{\Gamma_1^*(H)} I_1(\bar{\mathrm{t}}, H) I_1(\mathrm{t}, H)$$

just one sector

Same structure as in K-types

One Parameter Models

M-Type Hybrids

$$\Gamma_1(H) = \Gamma\left(1 - \frac{H}{2\pi i}\right)^4 \Gamma\left(\frac{1}{2} + \frac{H}{2\cdot 2\pi i}\right)^8 .$$

$$I_1(\mathtt{t}, H) = \frac{\Gamma\left(1 + \frac{H}{2\pi i}\right)^4}{\Gamma\left(\frac{H}{2\cdot 2\pi i} + \frac{1}{2}\right)^8} \sum_{a=0}^{\infty} e^{\mathtt{t}\left(\frac{H}{2\cdot 2\pi i} + a + \frac{1}{2} - q\right)} (-1)^{8a} \frac{\Gamma\left(a + \frac{H}{2\cdot 2\pi i} + \frac{1}{2}\right)^8}{\Gamma\left(1 + 2a + \frac{H}{2\pi i}\right)^4}$$

Two Parameter Models

Summary

- Procedure more involved
- **Structures match** one parameter cases
- **Conjectural expressions** for I function and gamma-class in **hybrid phases**
  - Annihilated by Picard-Fuchs operator

Section 4

Summary/Outlook

Summary

- We **proposed a general form of Z_{S^2}** in certain phases of GLSMs.
- We **checked our proposal** in 14 one parameter models and 2 two parameter models.
- Another way to write:

$$Z_{S^2} = \bar{I} M I.$$

  - I expanded in a basis of $\mathcal{H}$, M matrix
  - Form of M depends on the phase.
  - Related to CPT? [Cecotti,Vafa,91]

Outlook

- Extract **enumerative invariants** in hybrid phases.
- Clarify **relation to** results in **mathematics**
- **Pseudo-hybrids**:
  - similar structures
  - state space [Chiodo,Nagel,18]
- Hemisphere partition function in **hybrid models**:
  - Notion of **branes**
  - **Chern character** map

Section 5

Literature

Literature I

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