Grassmann flips and SODs

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joint w/ Ballard, Favero, McFaddin, Vandermolen

Outline

- Variation of GIT quotients and windows
- Q-construction
- Grassmann flips
- Semi Orthogonal Decompositions

Windows

VGIT and windows – Herbst-Hori-Page, Segal, Ballard-Favero-Katzarkov, Halpern-Leistner, . . .

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$$\Phi : D(X /\!\!/_G) \longrightarrow D([X/G]).$$

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Compare the derived categories of different GIT quotients by comparing the windows.

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When $G = \mathbb{G}_m = \operatorname{Spec} k[u, u^{-1}]$ and $X = \operatorname{Spec} T$, BDF defines

$$Q := \langle \pi(T), \sigma(T), u \rangle \subset k[\mathbb{G}_m \times X]$$

Q – partial compactification of the diagonal.

Example: If $X = \mathbb{A}^n$, $Q \cong \mathbb{A}^1 \times X$.

The Q-construction for $\mathbb{G}_m$-actions

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Theorem (Ballard-Diemer-Favero)

If $G = \mathbb{G}_m$ and X is smooth and affine,

$$\Phi_{Q_\pm} : D(X_\pm) \to D([X/\mathbb{G}_m])$$

is fully faithful.

If the Calabi-Yau condition is satisfied, this provides an equivalence $D(X_+) \cong D(X_-)$.

Example

VGIT problem: $X = \operatorname{Spec} \mathbb{C}[x_1, \cdots, x_n, y_1, \cdots, y_m] \cong \mathbb{C}^n \times \mathbb{C}^m$ with a $\mathbb{C}^*$-action such that

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$$X_+ = [(X \setminus V(\vec{x}))\, / \mathbb{C}^*] \cong \operatorname{tot}_{\mathbb{P}^{n-1}_{\vec{x}}} \mathcal{O}(-1)^{\oplus m}, \;\; X_- = [(X \setminus V(\vec{y}))\, / \mathbb{C}^*] \cong \operatorname{tot}_{\mathbb{P}^{m-1}_{\vec{y}}} \mathcal{O}(-1)^{\oplus n},$$

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$$\operatorname{Im} \Phi_{Q_+} = \langle \mathcal{O}(-n+1), \cdots, \mathcal{O}(-1), \mathcal{O} \rangle, \qquad \operatorname{Im} \Phi_{Q_-} = \langle \mathcal{O}, \mathcal{O}(1), \cdots, \mathcal{O}(m-1) \rangle,$$

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and a semi-orthogonal decomposition (assuming $n > m$)

$$D(X_+) = \langle \underbrace{D(k), \cdots, D(k)}_{n-m \text{ times}}, D(X_-) \rangle$$

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We get isomorphisms

$$X_+ \cong \mathrm{tot}_{\mathrm{Gr}(d,W)} \left(\mathcal{S}^{\oplus \dim W'} \right), \qquad X_- \cong \mathrm{tot}_{\mathrm{Gr}(d,W')} \left(\mathcal{S}^{\oplus \dim W} \right).$$

Generalized Q-construction

Let $\mathrm{GL}(V) = \operatorname{Spec} k[C, \det C^{-1}]$. Define

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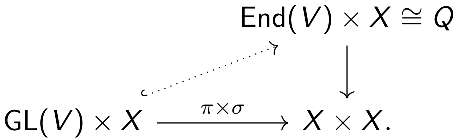

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$$\begin{array}{ccc} & & \mathrm{End}(V) \times X \cong Q \\ & \nearrow & \downarrow \\ \mathrm{GL}(V) \times X & \xrightarrow{\pi \times \sigma} & X \times X. \end{array}$$

Theorem (Ballard-C-Favero-McFaddin-Vandermolen)

Consider $Q_\pm := Q|_{X_\pm \times X}$. Then the functors

$$\Phi_{Q_\pm} : D(X_\pm) \to D([X/\mathrm{GL}(V)])$$

are fully-faithful, and the essential images are generated by Kapranov's collection.

Equivalences for Grassmann flops

Recall Kapranov's collection $\mathrm{K}_{d,\dim W}$ on $\mathrm{Gr}(d, W) = \left[\mathrm{Hom}(V, W)^{\mathrm{inj}}/\mathrm{GL}(V)\right]$:

$$\mathrm{K}_{d,\dim W} := \{L_\lambda(V) \,|\, \lambda \text{ is a Young diagram of height} \leq \dim W - d \text{ and width } \leq d\}$$

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Theorem (BCFMV)

Let $j : X_- \to [X/\mathrm{GL}(V)]$ denote the inclusion. When $\dim W = \dim W'$, the functor $j^* \circ \Phi_{Q_+} : D(X_+) \to D(X_-)$ is an equivalence.

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What about $\dim W \neq \dim W'$?

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Theorem (BCFMV)

There is a semi-orthogonal decomposition

$$D(X_+) = \langle \mathrm{O}_{d,\dim W-1}, D(X_-)\rangle,$$

where $O_{d,\dim W-1}$ *is supported on*

$$\left[\mathrm{Hom}(V, W)^{\mathrm{inj}} \times \{\textit{non-surjective maps}\}/\mathrm{GL}(V)\right] \subset X_+$$

Orthogonal category $\mathrm{O}_{d,s}$

Let H be a vector space of dimension $d-1$. Consider

$$h : \left[\frac{\mathrm{Hom}(V,W)^{\mathrm{inj}} \times \mathrm{Hom}(W',H) \times \mathrm{Hom}(H,V)^{\mathrm{inj}}}{\mathrm{GL}(H)\times \mathrm{GL}(V)}\right] \to \left[\frac{\mathrm{Hom}(V,W)^{\mathrm{inj}} \times \mathrm{Hom}(W',V)}{\mathrm{GL}(V)}\right].$$

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Definition

The category $\mathrm{O}_{d,s}$ is generated by

$$\left(h_*(L_{\lambda(i)})H^\vee\right)^\vee \otimes \det V^{s-i} \text{ for } \dim W' \leq i \leq s,$$

where $\lambda(i)$ runs over the set of Young diagrams of height $i+1-d$ and width $d-1$.

Example

Let $d = 2$, $\dim W' = 3$ and $\dim W = 4$. Then we have

$$X_+ \cong \mathrm{tot}_{\mathrm{Gr}(2,4)}\left(\mathcal{S}^{\oplus 3}\right), \qquad X_- \cong \mathrm{tot}_{\mathrm{Gr}(2,3)}\left(\mathcal{S}^{\oplus 4}\right).$$

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$$D(X_+) = \left\langle \mathcal{O}, \ydiagram{1}, \ydiagram{2}, \ydiagram{1,1}, \ydiagram{2,1}, \ydiagram{2,2} \right\rangle \qquad D(X_-) = \left\langle \mathcal{O}, \ydiagram{1}, \ydiagram{2} \right\rangle$$

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The orthogonal $\mathrm{O}_{2,3}$ is

$$\left\langle (h_*\mathcal{O})^\vee, \left(h_* L_{\ydiagram{1}}(H^\vee)\right)^\vee, \left(h_* L_{\ydiagram{1,1}}(H^\vee)\right)^\vee \right\rangle$$

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Thank you!