Quantum Sheaf Cohomology for Toric Complete Intersections

Zhentao Lu

BiCMR and Beijing Institute of Technology

Aug 18, 2020

GLSMs – 2020

Introduction

GLSM and quantum cohomology

Motivation: GLSM and quantum cohomology

- Twisted 2d GLSM, Witten (1993)
- (2,2) models and quantum cohomology: Batyrev (1993), Morrison-Plesser (1995), Szenes-Vergne (2004).
- (0,2) models and quantum sheaf cohomology (QSC): Adams-Basu-Sethi (2003), Katz-Sharpe (2006), McOrist-Melnikov (2008, 2009), Donagi et al. (2013, 2014).
- (0,2) mirror symmetry: Adams-Basu-Sethi (2003), Melnikov-Plesser (2011), Gu-Sharpe (2017).

An informal definition

Classical Sheaf Cohomology

- V : compact Kähler, $\dim V = n$.
- $\mathcal{E}$: holomorphic vector bundle over V, with $\wedge^{\mathrm{top}}\mathcal{E}^* \cong K_V$, $c_2(\mathcal{E}) = c_2(T_V)$.
- The *classical sheaf cohomology ring*

$$H^*_{\mathcal{E}}(V) := \oplus_{p,q} H^q(V, \wedge^p \mathcal{E}^*),$$

  with the product:
  $H^q(V, \wedge^p\mathcal{E}^*) \times H^{q'}(V, \wedge^{p'}\mathcal{E}^*) \xrightarrow{\cup} H^{q+q'}(V, \wedge^{p+p'}\mathcal{E}^*)$.
- In the (2,2) case, we have $\mathcal{E} = T_V$.

Classical Correlators

- The isomorphism $\wedge^{\mathrm{top}}\mathcal{E}^* \cong K_V$ induces $\phi : H^n(V, \wedge^{\mathrm{top}}\mathcal{E}^*) \cong H^n(V, K_V) \xrightarrow{\int_V} \mathbb{C}$.
- This enables us to define the *classical correlator*

$$\langle \sigma_1, ..., \sigma_s \rangle_0 := \bar{\phi}(\sigma_1 \cdot \sigma_2 \cdots \sigma_s),$$

  for $\sigma_i \in H^q(V, \wedge^q \mathcal{E}^*)$.
- In the (2,2) case: Dolbeault cohomology, Hodge theory.

Quantum Correlators

- For each effective curve β in $H_2(V, \mathbb{Z})$, one constructs a "suitable" moduli space of maps

$$f : \mathbb{P}^1 \to V, [f(\mathbb{P}^1)] = \beta.$$

- The *quantum correlator* is:

$$\langle \sigma_1, ..., \sigma_s \rangle := \sum_{\beta} \langle \tilde{\sigma}_1, ..., \tilde{\sigma}_s \rangle_{\beta} q^{\beta}.$$

- NLSM:
  - M_β Kontsevich moduli space of stable maps for the (2,2) case.
  - Difficulties for (0,2) case, Katz-Sharpe (2006).

Quantum Sheaf Cohomology

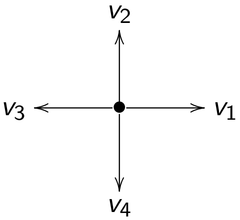
- Case by case, it is expected that induced sheaves can be constructed on GLSM style moduli spaces.
- Existing (0,2) GLSM theories: toric varieties and toric complete intersections, Grassmannians, flag varieties, toric stacks.
- Donagi et al. (2013, 2014) for toric case.
- I propose to extend the construction to toric complete intersections.

GLSM Style QSC and Correlators

QSC for Toric Varieties

The Toric Setting

- Donagi, Guffin, Katz, and Sharpe (2013, 2014)
- V: smooth projective toric variety, $\dim V = n$.
- Toric fan Σ, cones $\Sigma(k)$.
- Each ray $i \in \Sigma(1)$, corresponds to a toric divisor D_i.
- Cox ring $[x_1, ..., x_d]$, homogeneous coordinate ring.
- Example: $\mathbb{P}^1 \times \mathbb{P}^1$.



Toric Euler Sequence

- The cotangent bundle Ω_V fits in the toric Euler sequence

$$0 \to \Omega_V \to \oplus_{\rho \in \Sigma(1)} \mathcal{O}(-D_i) \xrightarrow{E_0} \mathcal{O} \otimes W \to 0,$$

where $W \cong H^2(V, \mathbb{C}) \cong \mathbb{C}^r$.

- For $\mathbb{P}^n$, this reduces to

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus(n+1)} \xrightarrow{E_0} \mathcal{O} \to 0.$$

- $\mathcal{E}^*$ is defined by the short exact sequence

$$0 \to \mathcal{E}^* \to \oplus \mathcal{O}(-D_i) \xrightarrow{E} \mathcal{O} \otimes W \to 0.$$

The classical sheaf cohomology

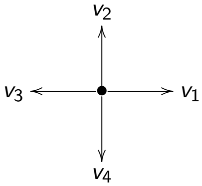
- The classical sheaf cohomology is

$$H^*_{\mathcal{E}}(V) \cong \mathrm{Sym}^* W / SR(V, \mathcal{E}),$$

where $SR(V, \mathcal{E})$ is the *Stanley-Reisner ideal for* $\mathcal{E}$ we define next.

- Special case: when $\mathcal{E} = T_V$, $E_0 = \sum_i x_i [D_i]$. let $\sigma_i = [D_i]$,

$$SR(V, T_V) = \langle \prod_{i \in K} \sigma_i | K \text{ is a primitive collection} \rangle.$$



Example: when $V = \mathbb{P}^1 \times \mathbb{P}^1$,

$$SR(V, T_V) = \langle \sigma_1^2, \sigma_2^2 \rangle.$$

The classical sheaf cohomology

- General case:
  $E \in \mathrm{Hom}(\oplus \mathcal{O}(-D_i), \mathcal{O} \otimes W) \cong H^0(\oplus \mathcal{O}(D_i)) \otimes W.$
  $$E_i = \sum_{\{j | D_j \sim D_i\}} a_{ij} x_j.$$
  Define $Q_c = \det(a_{ij})$, where $c = [D_i]$, and define
  $$Q_K = \prod_{c \in [K]} Q_c.$$
- Then we have
  $$SR(V, \mathcal{E}) = \langle Q_K | K \text{ is a primitive collection} \rangle.$$
- When E specializes to E_0 with $E_{0,i} = \sigma_i x_i$, we recover
  $$SR(V, T_V) = \langle \prod_{i \in K} \sigma_i | K \text{ is a primitive collection} \rangle.$$

The QSC Ring

Theorem (Donagi et al., 2014)

The QSC Ring takes the form

$$QH^*_{\mathcal{E}}(V) = (\mathrm{Sym}^* W \otimes \mathbb{C}[q^\beta])/QSR(X,\mathcal{E}), \text{ where}$$

$$QSR(X,\mathcal{E}) = \langle Q_K - q^{\beta_K} \prod_{c \in [K^-]} Q_c^{-d_c^{\beta_K}} | K \text{ is a primitive collection}\rangle.$$

Notations: β_K is an effective curve corresponds to K.

$$d_i^\beta = \langle \beta, D_i \rangle, [K^-] = \{i \in \Sigma(1) | d_i^{\beta_K} < 0\}.$$

Examples

- Take a basis $\beta_1, ..., \beta_r \in H_2(V, \mathbb{Z})$: $q^{\beta_j} \dashrightarrow q_j \in \mathbb{C}^*$.
  $QSR(V, \mathcal{E}) \dashrightarrow$ QSC Relations (QSCR).
  Example: For Hirzebruch surface F_n, the QSCR is

$$\begin{cases} \sigma_1^2 = q_1 \sigma_2^n \\ \sigma_2(\sigma_2 + n\sigma_1) = q_2 \end{cases} \quad \begin{cases} \sigma_1^2 \sigma_2^{-n} = q_1 \\ \sigma_2(\sigma_2 + n\sigma_1) = q_2 \end{cases}$$

The Quantum Correlator Formula

Theorem (L, arXiv:1511.09158)

Let V be a smooth, projective, nef-Fano toric variety, and $\mathcal{E}$ be a small deformation of the tangent bundle, then for $\sigma_i \in W \cong H^1(\mathcal{E}^*)$ and small $q = (q_1, ..., q_r)$, we have the correlator formula

$$\langle \sigma_1, ..., \sigma_s \rangle = \sum \frac{\sigma_1 \cdots \sigma_s}{\prod_c Q_c} \frac{\prod_j \tilde{v}_j}{\det(\tilde{v}_{j,k})},$$

where the summation is taken over the solutions to the QSC relations $\{\tilde{v}_j = q_j, j = 1, .., r\}$, with $\tilde{v}_j = \prod_c Q_c^{d_c^{\beta_j}} \in \mathrm{Sym}^* \mathrm{W}$.

Remark: Conjectured by McOrist-Melnikov (2008).

QSC for Toric Complete Intersections

The (2,2) case motivation

- (2,2) case (Intersection theory):

$$(D_1|_X, ..., D_s|_X)_X = (D_1, ..., D_s, [X])_V.$$

- Let X be a toric complete intersection in V and $\mathcal{E}_X$ be a deformation of the tangent bundle T_X.

  Interest: $H^q(X, \wedge^p \mathcal{E}_X^*)$, correlators.

For the toric part of $H^1(X, \mathcal{E}_X^*)$, we expect that the classical correlator can be computed by the following Sheaf COhomology REstriction (SCORE) formula:

$$\langle \sigma_1, ..., \sigma_s \rangle_{0,X} = \langle \sigma_1, ..., \sigma_s, [\mathcal{E}] \rangle_{0,V}$$

SCORE formula

Theorem (L)

Let X be a smooth toric complete intersection in V defined by $f_k \in H^0(V, \mathcal{O}(H_k)), k = 1, \ldots, m$ and $\mathcal{E}_X^*$ be a deformation of the cotangent bundle Ω_X defined by the middle cohomology of

$$\oplus \mathcal{O}_X(-H_k) \xrightarrow{J} \oplus \mathcal{O}_X(-D_i) \xrightarrow{E} \mathcal{O}_X \otimes W,$$

where $J = (J_1, \ldots, J_m)$ and

$$E \circ J_k = \gamma_k \cdot f_k \in \mathrm{Hom}(\mathcal{O}_V(-H_k), \mathcal{O}_V \otimes W).$$

Then we have a SCORE formula:

$$\langle \sigma_1, \sigma_2, \ldots, \sigma_s \rangle_{0,X} = \langle \sigma_1, \sigma_2, \ldots, \sigma_s, \gamma_1, \ldots, \gamma_m \rangle_{0,V}.$$

Description of the bundle $\mathcal{E}_X^*$

- Hypersurface case: $X \subset V$, $\mathcal{E}_X^*$ is the middle cohomology of

$$\mathcal{O}_X(-X) \xrightarrow{J} \oplus \mathcal{O}_X(-D_i) \xrightarrow{E} \mathcal{O}_X \otimes W \ , \text{ or}$$

$$0 \to \mathcal{O}_X(-X) \xrightarrow{J} \mathcal{E}_V^*|_X \to \mathcal{E}_X^* \to 0.$$

- Canonically $\wedge^n \mathcal{E}_V^*(X)|_X \cong \wedge^{n-1}\mathcal{E}_X^*$, hence

$$0 \to \wedge^n \mathcal{E}_V^* \to \wedge^n \mathcal{E}_V^*(X) \to \wedge^{n-1}\mathcal{E}_X^* \to 0.$$

(This is $0 \to \mathcal{O}_V \to \mathcal{O}_V(X) \to \mathcal{O}_V(X)|_X \to 0$ tensoring $\wedge^n \mathcal{E}_V^*$.)

- The (2,2) case: $J = (\partial f)$, $E = E_0$, $\mathcal{E}_X^* = \Omega_X$, and we have

$$0 \to \Omega_V^n \to \Omega_V^n(X) \xrightarrow{P.R.} \Omega_X^{n-1} \to 0.$$

Sequences for cohomology computation

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_X(-X) & = & \mathcal{O}_X(-X) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{E}_V^*|_X & \longrightarrow & \oplus\mathcal{O}_X(-D_i) & \longrightarrow & \mathcal{O}_X \otimes W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
0 & \longrightarrow & \mathcal{E}_X^* & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_X \otimes W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Technicality(1)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^n \mathcal{E}_V^* & \to & \wedge^n \mathcal{E}_V^*(X) & \to & \wedge^{n-1} \mathcal{E}_X^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \wedge^n(\oplus \mathcal{O}(-D_i)) & \to & \wedge^n(\oplus \mathcal{O}(-D_i))(X) & \to & \wedge^n(\oplus \mathcal{O}_X(-D_i))(X) & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & (\oplus \mathcal{O}(-D_i)) \otimes \mathrm{Sym}^{n-1} W & \to & (\oplus \mathcal{O}(-D_i))(X) \otimes \mathrm{Sym}^{n-1} W & \to & (\oplus \mathcal{O}_X(-D_i))(X) \otimes \mathrm{Sym}^{n-1} W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O} \otimes \mathrm{Sym}^n W & \to & \mathcal{O}(X) \otimes \mathrm{Sym}^n W & \to & \mathcal{O}_X(X) \otimes \mathrm{Sym}^n W & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Technicality(2)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \wedge^{n-1}\mathcal{E}_X^* & = & \wedge^{n-1}\mathcal{E}_X^* & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^{n-1}\mathcal{F} & \longrightarrow & \wedge^n(\oplus\mathcal{O}_X(-D_i))(X) & \longrightarrow & \wedge^n\mathcal{F}(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}\otimes\mathrm{Sym}^{n-2}W & \longrightarrow & \wedge^2(\oplus\mathcal{O}_X(-D_i))(X)\otimes\mathrm{Sym}^{n-2}W & \longrightarrow & \wedge^2\mathcal{F}\otimes\mathrm{Sym}^{n-2}W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}\otimes\mathrm{Sym}^{n-1}W & \longrightarrow & \oplus\mathcal{O}_X(-D_i)(X)\otimes\mathrm{Sym}^{n-1}W & \longrightarrow & \mathcal{F}\otimes\mathrm{Sym}^{n-1}W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathcal{O}_X\otimes\mathrm{Sym}^nW & = & \mathcal{O}_X\otimes\mathrm{Sym}^nW & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

The classical sheaf cohomology $H^*_{\mathcal{E}}(X)^{\text{toric}}$

- We would like to relate the map

$$H^1(\mathcal{E}_X^*) \times \ldots \times H^1(\mathcal{E}_X^*) \to H^{n-1}(\wedge^{n-1}\mathcal{E}_X^*) \cong \mathbb{C}$$

to the toric ambient spaces maps:

$$\begin{array}{ccccc} \mathrm{Sym}^{n-1}W & \longrightarrow & H^0(S_1(X)|_X) & \longrightarrow & \mathrm{Sym}^n W \\ \downarrow & & \downarrow & & \downarrow \\ H^{n-1}(\wedge^{n-1}\mathcal{E}_X^*) & = & H^{n-1}(\wedge^{n-1}\mathcal{E}_X^*) & \xrightarrow{\delta} & H^n(\wedge^n\mathcal{E}_V^*), \end{array}$$

where $S_1(X)|_X$ is the kernel of

$$\oplus\mathcal{O}_X(-D_i) \otimes \mathrm{Sym}^{n-1}W \to \mathcal{O}_X(X) \otimes \mathrm{Sym}^n W.$$

SCORE formula

Theorem (L)

Let X be a smooth toric complete intersection in V defined by $f_k \in H^0(V, \mathcal{O}(H_k)), k = 1, ..., m$ and $\mathcal{E}_X^*$ be a deformation of the cotangent bundle Ω_X defined by the middle cohomology of

$$\oplus \mathcal{O}_X(-H_k) \xrightarrow{J} \oplus \mathcal{O}_X(-D_i) \xrightarrow{E} \mathcal{O}_X \otimes W,$$

where $J = (J_1, ..., J_m)$ and

$$E \circ J_k = \gamma_k \cdot f_k \in \mathrm{Hom}(\mathcal{O}_V(-H_k), \mathcal{O}_V \otimes W).$$

Then we have a SCORE formula:

$$\langle \sigma_1, \sigma_2, ..., \sigma_s \rangle_{0,X} = \langle \sigma_1, \sigma_2, ..., \sigma_s, \gamma_1, ..., \gamma_m \rangle_{0,V}.$$

Morrison-Plesser moduli spaces

- Morrison-Plesser (1995), Batyrev-Materov (2002).
- Quasimap: Ciocan-Fontanine et al. (2014)

  V_β: toric

  $X_\beta \subset V_\beta$: not necessarily toric complete intersection.
- Induced sheaf: E, J.

An Example

- $V = \mathbb{P}^2 : [x_0 : x_1 : x_2]$, $\beta = [D_1]$, $X = (f), f = x_0^3 + x_1^3 + x_2^3$
- $\phi : \mathbb{P}^1 \dashrightarrow V$

  $[t_0 : t_1] \mapsto [a_0 t_0 + a_1 t_1 : b_0 t_0 + b_1 t_1 : c_0 t_0 + c_1 t_1]$

  $V_\beta = \mathbb{P}^5 : [a_0 : a_1 : b_0 : b_1 : c_0 : c_1]$
- $\phi : \mathbb{P}^1 \dashrightarrow X$

  $f \circ \phi : (a_0 t_0 + a_1 t_1)^3 + (b_0 t_0 + b_1 t_1)^3 + (c_0 t_0 + c_1 t_1)^3 = 0$

  $$\Rightarrow \begin{cases} a_0^3 + b_0^3 + c_0^3 = 0 \\ a_0^2 a_1 + b_0^2 b_1 + c_0^2 c_1 = 0 \\ a_0 a_1^2 + b_0 b_1^2 + c_0 c_1^2 = 0 \\ a_1^3 + b_1^3 + c_1^3 = 0 \end{cases}.$$

  X_β is NOT a complete intersection.

Quantum restrictions

- Definition of the quantum correlator:

$$\langle \sigma_1, \ldots, \sigma_s \rangle_X := \sum_{\beta} (-1)^{\beta} \langle \sigma_1, \ldots, \sigma_s \rangle_{\beta, X} q^{\beta}$$

- Naive quantum restriction:

$$\langle \sigma_1, \ldots, \sigma_s \rangle_X = \sum_{\beta} (-1)^{\beta} \langle \sigma_1, \ldots, \sigma_s, \gamma^{n_\beta} \rangle_{\beta, V}$$

- Calabi-Yau hypersurface case (McOrist-Melnikov):

$$\langle \sigma_1, \ldots, \sigma_s \rangle_X := \langle \sigma_1, \ldots, \sigma_s, \frac{\gamma}{1+\gamma} \rangle_V$$

Concluding Remarks

Concluding Remarks

- QSC helps us to compute correlators in geometric settings.
- There are both special cases and general constructions "ready" to be carried out.
- NLSM style QSC is yet to be constructed.
- Higher rank bundles and (0,2) heterotic mirror symmetry.
- Frobenius structures.
- Please send comments and suggestions to zhentao@sas.upenn.edu