GLSMs for Symplectic Grassmannians

Hao Zou

Department of Physics, Virginia Tech

Joint work with Wei Gu and Eric Sharpe, arXiv:2008.02281

In this talk:

- Review of GLSMs for ordinary Grassmannians
- The symplectic Grassmannians and its GLSM realization
- Discussion of phases of this GLSM

Review of GLSMs for Grassmannians

[Witten, hep-th/9301042 & hep-th/9312104]

The GLSM for the Grassmannian, $G(k,n)$, is a two dimensional $\mathcal{N}=2\ U(k)$ gauge theory with n fundamentals, Φ_i^a.

In the infrared, for nonzero FI parameter r, vanishing scalar potential forces

$$\frac{1}{e^2}D_a^b = \sum_{i=1}^{n} \bar{\phi}_a^i \phi_i^b - r\delta_a^b = 0$$

- For $r \gg 0$, $\{D=0\}$ defines orthonormal conditions for k vectors in $\mathbb{C}^n$, so the space of classical vacua is $G(k,n)$.
- For $r \ll 0$, only Coulomb vacua.

Review of GLSMs for Grassmannians

[Morrison & Plesser, hep-th/9412236; Hori & Tong, hep-th/0609032]

Coulomb branch:

$U(k)$ Higgsed to $U(1)^k$, diagonal σ survive.

Excluded locus:

$$\sigma_a \neq 0, \quad \sigma_a \neq \sigma_b \quad \text{if } a \neq b.$$

One-loop twisted effective superpotential:

$$\widetilde{W}_{\text{eff}} = - t \sum_{a=1}^{k} \Sigma_a - \sum_{i=1}^{n} \sum_{c=1}^{k} \rho^a_{ic} \Sigma_a \left[\ln \left(\rho^b_{ic} \Sigma_b \right) - 1 \right] - \sum_{\mu \neq \nu} \alpha^a_{\mu\nu} \Sigma_a \left[\ln \left(\alpha^b_{\mu\nu} \Sigma_b \right) - 1 \right]$$

with $\rho^a_{ic} = \delta^a_c$ and $\alpha^a_{\mu\nu} = -\delta^a_\mu + \delta^a_\nu$.

Review of GLSMs for Grassmannians

Chiral ring relations:

$$\exp\left(\frac{\partial \widetilde{W}_{\text{eff}}}{\partial \sigma_a}\right) = 1 \quad \Rightarrow \quad \sigma_a^n = (-)^{k-1} q$$

[Witten, hep-th/9312104; Morrison & Plesser, hep-th/9412236]

These chiral ring relations imply the quantum cohomology ring relations of $G(k,n)$. The σ_a's are interpreted as Chern roots of $\mathcal{S}^*$, where $\mathcal{S}$ is the tautological bundle:

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{V} \longrightarrow \mathcal{Q} \longrightarrow 0$$

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Next let us consider the symplectic Grassmannians ...

Symplectic Grassmannians

Ordinary Grassmannians can be described mathematically as cosets $SL(n,\mathbb{C})/P$, and as they have global symmetry $SU(n)$, are called type A Grassmannians.

There are also analogous homogeneous spaces of the form $SO(2n+1,\mathbb{C})/P$, $Sp(2n,\mathbb{C})/P$ and $SO(2n,\mathbb{C})/P$, called Grassmannians of type B, C, D.

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In this talk, let us focus on type C Grassmannians, which are cosets $Sp(2n,\mathbb{C})/P$, and are called the *Symplectic Grassmannians*.

Definition

Given a symplectic form ω in $\mathbb{C}^{2n}$, the *symplectic Grassmannian* $SG(k,2n)$ is the space parameterizing k-dim'l ($k \leq n$) subspaces in $\mathbb{C}^{2n}$ which are isotropic with respect to ω. When k is maximal, it is also called the *Lagrangian Grassmannian*.

GLSM for Symplectic Grassmannian

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GLSM description:
The GLSM for $SG(k, 2n)$ is an $U(k)$ gauge theory defined by

- $2n$ chiral multiplets $\Phi^a_{\pm i}$ in the fundamental representation V,
- 1 chiral multiplet P_{ab} in the anti-symmetric tensor representation $\wedge^2 V^*$,

with superpotential

$$W = \sum_{\alpha,\beta,a,b} P_{ab}\Phi^a_\alpha \Phi^b_\beta \omega^{\alpha\beta} = \sum_{i,a,b} P_{ab}\Phi^a_i \Phi^b_{-i}, \quad \text{for} \quad \omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

GLSM for Symplectic Grassmannian

In the infrared, the vacuum configuration on the Higgs branch ($r \neq 0$ and $\sigma = 0$) is determined by the scalar potential.

First, vanishing potential requires

$$\frac{1}{e^2} D_a^b = \sum_{i=1}^{n} \left(\bar{\phi}_a^i \phi_i^b + \bar{\phi}_a^{-i} \phi_{-i}^b \right) - 2\bar{p}^{bc} p_{ac} - r\delta_a^b = 0,$$

$$F_{\pm i}^a = \frac{\partial W}{\partial \phi_{\pm i}^a} = 0, \quad F^{ab} = \frac{\partial W}{\partial p_{ab}} = 0.$$

When $r \gg 0$, D-term conditions determine the ambient space to be $G(k, 2n)$. F-term conditions restrict to subspace satisfying the following isotropy condition

$$\sum_{i=1}^{n} \left(\phi_i^a \phi_{-i}^b - \phi_i^b \phi_{-i}^a \right) = 0.$$

Global symmetries

Recall: the symplectic Grassmannian $SG(k, 2n)$ can also be defined as the coset

$$Sp(2n, \mathbb{C})/P.$$

Correspondingly, its GLSM should have the global symmetry $Sp(2n)$.

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Check:

- Rotations of $2n$ chiral fields $\Rightarrow U(2n)$.
- Invariant of the superpotential requires to preserve the symplectic form $\Rightarrow Sp(2n, \mathbb{C})$.

Therefore, the global symmetry is $Sp(2n)$.

GLSM for Symplectic Grassmannian

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$$p_{ab} = \begin{bmatrix} 0 & * & \cdots & 0 \\ -* & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The last entry corresponds to following diagonal D-term:

$$\frac{1}{e^2} D_k^k = \sum_{i=1}^{n} \left(\left| \phi_i^k \right|^2 + \left| \phi_{-i}^k \right|^2 \right) - r = 0.$$

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There are no classical Higgs vacua in $r \ll 0$ phase when k is odd; pure Coulomb branch

GLSM for Symplectic Grassmannian

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What about the $r \ll 0$ phase?

For k even, there is a nontrivial Higgs branch.

For example, when $k = 2$, $U(2)$ Higgsed to $SU(2)$ due to $\langle P_{12} \rangle$ and so this phase is actually an $SU(2)$ gauge theory with $2n$ fundamentals.

This theory has been well studied and the Witten index for this theory has been given:

$$\frac{1}{2}(2n - 2) = n - 1.$$

[Hori & Tong, hep-th/0609032;

Benini, Eager, Hori & Tachikawa, arXiv:1308.4896[hep-th]]

Later we will see this is consistent with Witten indices.

GLSM for Symplectic Grassmannian

On the Coulomb branch, the generic coordinates σ_a's should satisfy

$$\sigma_a \neq 0, \quad \sigma_a \pm \sigma_b \neq 0 \quad \text{if } a \neq b.$$

The effective twisted superpotential is

$$\begin{aligned}
\widetilde{W}_{\text{eff}} = & -t \sum_{a=1}^{n} \Sigma_a - \sum_{i=1}^{2n} \sum_{a,b=1}^{n} \rho_{ia}^{b} \Sigma_b \left[\ln \left(\sum_{b=1}^{n} \rho_{ia}^{c} \Sigma_c \right) - 1 \right] \\
& - \sum_{\mu>\nu=1}^{n} \sum_{a=1}^{n} \rho_{\mu\nu}^{a} \Sigma_a \left[\ln \left(\sum_{b=1}^{n} p_{\mu\nu}^{b} \Sigma_b \right) - 1 \right] \\
& - \sum_{\mu\neq\nu=1}^{n} \sum_{a=1}^{n} \alpha_{\mu\nu}^{a} \Sigma_a \left[\ln \left(\sum_{b=1}^{n} \alpha_{\mu\nu}^{b} \Sigma_b \right) - 1 \right],
\end{aligned}$$

with $\rho_{ib}^{a} = \delta_{b}^{a}$, $\rho_{\mu\nu}^{a} = -\delta_{\mu}^{a} - \delta_{\nu}^{a}$ and $\alpha_{\mu\nu}^{a} = -\delta_{\mu}^{a} + \delta_{\nu}^{a}$.

GLSM for Symplectic Grassmannian

Chiral ring relations:

$$\exp\left(\frac{\partial \widetilde{W}}{\partial \sigma_a}\right) = 1 \quad \Rightarrow \quad q \prod_{b \neq a} (\sigma_a + \sigma_b) = \sigma_a^{2n}$$

- When $k = n$, we can show that these chiral ring relations do match the quantum cohomology ring relations. ‣ Explain

  [Buch, Kresch & Tamvakis, arXiv:0809.4966[math-AG]]

- When $k < n$, we have checked in special cases that these chiral ring relations do reproduce the quantum cohomology ring relations.
- When k is odd, it can be argued that the number of Coulomb vacua matches the $\chi(SG(k, 2n)) = 2^k \binom{n}{k}$. ‣ Explain
- When k is even, several examples have been checked and results are consistent with Witten indices. ‣ Explain

Calabi-Yau conditions

Known result in math:
The intersection of the Plücker embedding of $SG(k,2n)$ with a hypersurface of degree $2n-k+1$ is a Calabi-Yau.

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Known result in math:
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In the GLSM for $SG(k, 2n)$, under an $U(1) \subset U(k)$,

- $2n$ chirals in the $U(k)$ fundamental V contribute $2n$,
- 1 chiral in $\wedge^2 V^*$ contributes $-(k-1)$,

so the sum of $U(1)$ charges is $2n - k + 1$.
After the Plücker embedding of $SG(k, 2n)$, the sum of the same $U(1)$ charges is still $2n - k + 1$.

Therefore, to build a Calabi-Yau, we need to intersect the image of $SG(k, 2n)$ with a hypersurface of degree $2n - k + 1$.

Outline of the GLSM for Orthogonal Grassmannian

Definition

The *orthogonal Grassmannian* $OG(k, n)$ is the space parameterizing the k-dim'l subspaces in $\mathbb{C}^n$ which are isotropic w.r.t. a given symmetric bilinear form g.

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The *orthogonal Grassmannian* $OG(k,n)$ is the space parameterizing the k-dim'l subspaces in $\mathbb{C}^n$ which are isotropic w.r.t. a given symmetric bilinear form g.

The GLSM for $OG(k,n)$ is an $U(k)$ gauge theory having:

- n chiral multiplets Φ^a_α in the fundamental representation V
- 1 chiral multiplet Q_{ab} in the symmetric tensor representation $\mathrm{Sym}^2 V^*$

with superpotential

$$W = \sum_{\alpha,\beta,a,b} Q_{ab}\Phi^a_\alpha \Phi^b_\beta g^{\alpha\beta}.$$

Additional results

- Equivariant quantum cohomology rings can also be duplicated from physics
- Orthogonal Grassmannians $OG(k, 2n)$ and $OG(k, 2n+1)$
- Mirrors of SG and OG
- Quantum K-theory for symplectic Grassmannians

[Gu, Mihalcea, Sharpe, HZ, arXiv:2008.04909 [hep-th]]

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Future work

- Isotropic flag varieties, including G_2 flags [In progress]

Thank you!

Quantum cohomology ring relations

Consider $SG(n, 2n)$ with $n = 2k + 1$ odd. We can rewrite the chiral ring relations as following:

$$q\sigma_a^{2k} + qe_2(\sigma)\sigma_a^{2k-2} + \cdots + qe_{2k-2}(\sigma)\sigma_a^2 + qe_{2k}(\sigma) \;=\; \sigma_a^{4k+2},$$

where $e_i(\sigma)$ is the i-th elementary symmetric polynomial.

Since the $e_i(\sigma)$ are Weyl invariant, the $e_i(\sigma)$ are constant on Weyl orbits. Rewrite the above equation as

$$\begin{aligned} P(\sigma_a^2) \;&\equiv\; (\sigma_a^2)^{2k+1} - q(\sigma_a^2)^k - \cdots - qe_{2k-2}(\sigma)\sigma_a^2 - qe_{2k}(\sigma) \\ &=\; 0, \end{aligned}$$

Then $\{\sigma_a^2\}$ should satisfy relations determined by the cofficients of $P(\sigma_a^2)$ according to *Vieta's formula*.

Quantum cohomology ring relations

Vieta's formula tell us that for the polynomial:

$$P(x) = (x)^{2k+1} - q(x)^k - \cdots - qe_{2k-2}(\sigma)x^2 - qe_{2k}(\sigma),$$
$$\equiv (x - x_1)(x - x_2)\cdots(x - x_{2k+1}),$$

then its $(2k+1)$ roots $\{x_a\}$ should satisfy following relations

$$\sum_{1\le a_1<\cdots<a_\ell\le n} x_{a_1}\cdots x_{a_\ell} \;=\; (-)^{\ell-1} e_{2\ell-n-1}(\sigma) q,$$

for $\ell \le n = 2k+1$ and we've used $e_i = 0$ for $i < 0$ and $e_0 = 1$.

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for $\ell \leq n = 2k+1$ and we've used $e_i = 0$ for $i < 0$ and $e_0 = 1$.

$x_a \to \sigma_a^2 \quad \Rightarrow$ It recovers the quantum cohomology ring relations for $SG(n, 2n)$ when $k = 2k+1$.

[Buch, Kresch & Tamvakis, arXiv:0809.4966[math-AG]]

Similar analysis applies to $SG(n, 2n)$ for n even cases.

Back

Witten indices for $SG(k, 2n)$ with k odd

For k odd, as discussed before, there are only the geometric phase and the pure Coulomb branch phase. Therefore, the number of Coulomb vacua should match the Euler characteristic of $SG(k, 2n)$:

$$2^k \binom{n}{k}.$$

The chiral ring relations,

$$q \prod_{b \neq a} (\sigma_a + \sigma_b) = \sigma_a^{2n},$$

which can also be rewritten as

$$\sigma_a^{2n} - q \left[\sigma_a^{k-1} + e_2(\sigma)\sigma_a^{k-3} + \cdots + e_{k-1}(\sigma) \right] \; = \; 0.$$

Witten indices for $SG(k, 2n)$ with k odd

Since k is odd and we count solutions on the excluded locus

$$\sigma_a \neq 0, \quad \sigma_a \neq \pm\sigma_b \text{ for } a \neq b,$$

there is a $\mathbb{Z}_2$-symmetry among solutions:

If $\{\sigma_1, \ldots, \sigma_k\}$ is one solution, then $\{-\sigma_1, \ldots, -\sigma_k\}$ is another solution.

Further, all solutions satisfy the same equaitons due to the Weyl symmetry. So we can count the number of solutions as following:

$$\frac{2n(2n-2)\cdots(2n-2k+2)}{k!} = 2^k \binom{n}{k}.$$

Go Back

Witten indices for $SG(k, 2n)$ with k even

For k even, the $r \ll 0$ phase is a mixed Higgs-Coulomb branch, so the total Witten index consist of contributions from both Higgs and Coulomb branches. This total number should match the Euler characteristic of $SG(k, 2n)$.

Let us check in the $k = 2$ case. On the Coulomb branch, when $k = 2$, the chiral ring relations are

$$q(\sigma_1 + \sigma_2) = \sigma_1^{2n}, \quad q(\sigma_1 + \sigma_2) = \sigma_2^{2n},$$

which imply $\sigma_1^{2n} = \sigma_2^{2n}$. Exclude $\sigma_1 = \pm\sigma_2$, there are $(2n - 2)$ relations between σ_1 and σ_2. Then the chiral ring relation will reduce to an equation of degree $(2n - 1)$, which has $(2n - 1)$ solutions.

Witten indices for $SG(k, 2n)$ with k even

Therefore, the number of Coulomb vacua is

$$\frac{1}{2}(2n-2)(2n-1) = (2n-1)(n-1).$$

The Higgs branch in $r \ll 0$ is a $SU(2)$ gauge theory with $2n$ fundamentals, which has Witten index $(n-1)$.

So the total number is

$$(n-1) + (2n-1)(n-1) = 2^2 \binom{n}{2},$$

which matches the Euler characteristic of $SG(2, 2n)$.

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