GLSMs for Symplectic Grassmannians

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In this talk:

- Review of GLSMs for ordinary Grassmannians
- The symplectic Grassmannians and its GLSM realization
- Discussion of phases of this GLSM
Review of GLSMs for Grassmannians

[Witten, hep-th/9301042 & hep-th/9312104]

The GLSM for the Grassmannian, $G(k, n)$, is a two dimensional $\mathcal{N} = 2$ $U(k)$ gauge theory with $n$ fundamentals, $\Phi^a_i$.

In the infrared, for nonzero FI parameter $r$, vanishing scalar potential forces

$$\frac{1}{e^2} D_a = \sum_{i=1}^{n} \bar{\phi}^a_i \phi^b_i - r \delta^b_a = 0$$

- For $r \gg 0$, $\{D = 0\}$ defines orthonormal conditions for $k$ vectors in $\mathbb{C}^n$, so the space of classical vacua is $G(k, n)$.
- For $r \ll 0$, only Coulomb vacua.
Review of GLSMs for Grassmannians

[Morrison & Plesser, hep-th/9412236; Hori & Tong, hep-th/0609032]

Coulomb branch:

\[ U(k) \text{ Higgsed to } U(1)^k, \text{ diagonal } \sigma \text{ survive.} \]

Excluded locus:

\[ \sigma_a \neq 0, \quad \sigma_a \neq \sigma_b \quad \text{if } a \neq b. \]

One-loop twisted effective superpotential:

\[
\tilde{W}_{\text{eff}} = -t \sum_{a=1}^{k} \Sigma_a - \sum_{i=1}^{n} \sum_{c=1}^{k} \rho_{ic}^a \Sigma_a \left[ \ln \left( \rho_{ic}^a \Sigma_b \right) - 1 \right] \\
- \sum_{\mu \neq \nu} \alpha_{\mu \nu}^a \Sigma_a \left[ \ln \left( \alpha_{\mu \nu}^b \Sigma_b \right) - 1 \right]
\]

with \( \rho_{ic}^a = \delta_c^a \) and \( \alpha_{\mu \nu}^a = -\delta^a_{\mu} + \delta^a_{\nu}. \)
Review of GLSMs for Grassmannians

Chiral ring relations:

\[
\exp \left( \frac{\partial \tilde{W}_{\text{eff}}}{\partial \sigma_a} \right) = 1 \quad \Rightarrow \quad \sigma_a^n = (-)^{k-1} q
\]

[Witten, hep-th/9312104; Morrison & Plesser, hep-th/9412236]

These chiral ring relations imply the quantum cohomology ring relations of $G(k, n)$. The $\sigma_a$'s are interpreted as Chern roots of $S^*$, where $S$ is the tautological bundle:

\[
0 \longrightarrow S \longrightarrow V \longrightarrow Q \longrightarrow 0
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Chiral ring relations:

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\end{array}
\]

Next let us consider the symplectic Grassmannians ...
Symplectic Grassmannians

Ordinary Grassmannians can be described mathematically as cosets $SL(n, \mathbb{C})/P$, and as they have global symmetry $SU(n)$, are called type A Grassmannians.

There are also analogous homogeneous spaces of the form $SO(2n + 1, \mathbb{C})/P$, $Sp(2n, \mathbb{C})/P$ and $SO(2n, \mathbb{C})/P$, called Grassmannians of type B, C, D.
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In this talk, let us focus on type C Grassmannians, which are cosets $Sp(2n, \mathbb{C})/P$, and are called the Symplectic Grassmannians.

Definition

Given a symplectic form $\omega$ in $\mathbb{C}^{2n}$, the symplectic Grassmannian $SG(k, 2n)$ is the space parameterizing $k$-dim’l ($k \leq n$) subspaces in $\mathbb{C}^{2n}$ which are isotropic with respect to $\omega$. When $k$ is maximal, it is also called the Lagrangian Grassmannian.
GLSM for Symplectic Grassmannian

Alternatively,

**Definition**

$SG(k, 2n)$ is the zero locus in $G(k, 2n)$ of a global section of the vector bundle $\wedge^2(S^*)$. 
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**Definition**

\( SG(k, 2n) \) is the zero locus in \( G(k, 2n) \) of a global section of the vector bundle \( \wedge^2(S^*) \).

GLSM description:
The GLSM for \( SG(k, 2n) \) is an \( U(k) \) gauge theory defined by
- 2n chiral multiplets \( \Phi_{\pm i}^a \) in the fundamental representation \( V \),
- 1 chiral multiplet \( P_{ab} \) in the anti-symmetric tensor representation \( \wedge^2 V^* \),

with superpotential

\[
W = \sum_{\alpha, \beta, a, b} P_{ab} \Phi_{\alpha}^a \Phi_{\beta}^b \omega^{\alpha \beta} = \sum_{i, a, b} P_{ab} \Phi_i^a \Phi_{-i}^b, \quad \text{for} \quad \omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]
GLSM for Symplectic Grassmannian

In the infrared, the vacuum configuration on the Higgs branch \((r \neq 0 \text{ and } \sigma = 0)\) is determined by the scalar potential.

First, vanishing potential requires

\[
\frac{1}{e^2} D^b_a = \sum_{i=1}^{n} \left( \bar{\phi}^i a \phi^b_i + \bar{\phi}^{-i} a \phi^b_{-i} \right) - 2p^{bc} \bar{p}_{ac} - r \delta^b_a = 0,
\]

\[
F^a_{\pm i} = \frac{\partial W}{\partial \phi^a_{\pm i}} = 0, \quad F^{ab} = \frac{\partial W}{\partial p_{ab}} = 0.
\]

When \(r \gg 0\), \(D\)-term conditions determine the ambient space to be \(G(k, 2n)\). \(F\)-term conditions restrict to subspace satisfying the following isotropy condition

\[
\sum_{i=1}^{n} \left( \phi^a_i \phi^b_{-i} - \phi^b_i \phi^a_{-i} \right) = 0.
\]
Global symmetries

Recall: the symplectic Grassmannian $SG(k, 2n)$ can also be defined as the coset

$$Sp(2n, \mathbb{C})/P.$$ 

Correspondingly, its GLSM should have the global symmetry $Sp(2n)$. 
Global symmetries

Recall: the symplectic Grassmannian $SG(k, 2n)$ can also be defined as the coset

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Check:

- Rotations of $2n$ chiral fields $\Rightarrow U(2n)$.
- Invariant of the superpotential requires to preserve the symplectic form $\Rightarrow Sp(2n, \mathbb{C})$.

Therefore, the global symmetry is $Sp(2n)$. 
GLSM for Symplectic Grassmannian

What about the \( r \ll 0 \) phase?

The simpler case is when \( k \) is odd. Note that \( p_{ab} \) is antisymmetric, we can diagonalize it in terms of \( 2 \times 2 \) blocks.

\[
p_{ab} = \begin{pmatrix}
0 & * & \cdots & 0 \\
- * & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{pmatrix}
\]

The last entry corresponds to following diagonal \( D \)-term:

\[
e_{2k}^D = \sum_{i=1}^{n} (|\phi_{k_i}|^2 + |\phi_{k_i-1}|^2) - r = 0.
\]

There are no classical Higgs vacua in the \( r \ll 0 \) phase when \( k \) is odd; pure Coulomb branch.
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What about the $r \ll 0$ phase?

For $k$ even, there is a nontrivial Higgs branch.

For example, when $k = 2$, $U(2)$ Higgsed to $SU(2)$ due to $\langle P_{12} \rangle$ and so this phase is actually an $SU(2)$ gauge theory with $2n$ fundamentals.

This theory has been well studied and the Witten index for this theory has been given:

$$\frac{1}{2} (2n - 2) = n - 1.$$  

[Hori & Tong, hep-th/0609032; Benini, Eager, Hori & Tachikawa, arXiv:1308.4896[hep-th]]

Later we will see this is consistent with Witten indices.
On the Coulomb branch, the generic coordinates $\sigma_a$’s should satisfy

$$\sigma_a \neq 0, \quad \sigma_a \pm \sigma_b \neq 0 \quad \text{if} \ a \neq b.$$

The effective twisted superpotential is

$$\tilde{W}_{\text{eff}} = -t \sum_{a=1}^{n} \Sigma_a - \sum_{i=1}^{2n} \sum_{a,b=1}^{n} \rho^b_{ia} \Sigma_b \left[ \ln \left( \sum_{b=1}^{n} \rho^c_{ia} \Sigma_c \right) - 1 \right]$$

$$- \sum_{\mu > \nu = 1}^{n} \sum_{a=1}^{n} \rho^a_{\mu \nu} \Sigma_a \left[ \ln \left( \sum_{b=1}^{n} p^b_{\mu \nu} \Sigma_b \right) - 1 \right]$$

$$- \sum_{\mu \neq \nu = 1}^{n} \sum_{a=1}^{n} \alpha^a_{\mu \nu} \Sigma_a \left[ \ln \left( \sum_{b=1}^{n} \alpha^b_{\mu \nu} \Sigma_b \right) - 1 \right],$$

with $\rho^a_{ib} = \delta^a_b$, $\rho^a_{\mu \nu} = -\delta^a_{\mu} - \delta^a_{\nu}$ and $\alpha^a_{\mu \nu} = -\delta^a_{\mu} + \delta^a_{\nu}$. 
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Chiral ring relations:

\[ \exp \left( \frac{\partial \tilde{W}}{\partial \sigma_a} \right) = 1 \quad \Rightarrow \quad q \prod_{b \neq a} (\sigma_a + \sigma_b) = \sigma_a^{2n} \]

- When \( k = n \), we can show that these chiral ring relations do match the quantum cohomology ring relations. [Buch, Kresch & Tamvakis, arXiv:0809.4966[math-AG]]

- When \( k < n \), we have checked in special cases that these chiral ring relations do reproduce the quantum cohomology ring relations.

- When \( k \) is odd, it can be argued that the number of Coulomb vacua matches the \( \chi(SG(k, 2n)) = 2^k \binom{n}{k} \).

- When \( k \) is even, several examples have been checked and results are consistent with Witten indices.
Calabi-Yau conditions

Known result in math:
The intersection of the Plücker embedding of $SG(k, 2n)$ with a hypersurface of degree $2n - k + 1$ is a Calabi-Yau.
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The intersection of the Plücker embedding of $SG(k, 2n)$ with a hypersurface of degree $2n - k + 1$ is a Calabi-Yau.

In GLSM, the Calabi-Yau conditions are that the sum of the charges under any $U(1)$ subgroup of the gauge group vanishes.

In the GLSM for $SG(k, 2n)$, under an $U(1) \subset U(k)$,
- $2n$ chirals in the $U(k)$ fundamental $V$ contribute $2n$,
- $1$ chiral in $\wedge^2 V^*$ contributes $-(k - 1)$,
so the sum of $U(1)$ charges is $2n - k + 1$.

After the Plücker embedding of $SG(k, 2n)$, the sum of the same $U(1)$ charges is still $2n - k + 1$.

Therefore, to build a Calabi-Yau, we need to intersect the image of $SG(k, 2n)$ with a hypersurface of degree $2n - k + 1$. 
Outline of the GLSM for Orthogonal Grassmannian

Definition

The orthogonal Grassmannian $OG(k, n)$ is the space parameterizing the $k$-dim'l subspaces in $\mathbb{C}^n$ which are isotropic w.r.t. a given symmetric bilinear form $g$. 
Outline of the GLSM for Orthogonal Grassmannian

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The *orthogonal Grassmannian* \( OG(k, n) \) is the space parameterizing the \( k \)-dim'l subspaces in \( \mathbb{C}^n \) which are isotropic w.r.t. a given symmetric bilinear form \( g \).

The GLSM for \( OG(k, n) \) is an \( U(k) \) gauge theory having:

- \( n \) chiral multiplets \( \Phi^a_\alpha \) in the fundamental representation \( V \)
- 1 chiral multiplet \( Q_{ab} \) in the symmetric tensor representation \( \text{Sym}^2 V^* \)

with superpotential

\[
W = \sum_{\alpha, \beta, a, b} Q_{ab} \Phi^a_\alpha \Phi^b_\beta g^{\alpha\beta}.
\]
Additional results

- Equivariant quantum cohomology rings can also be duplicated from physics
- Orthogonal Grassmannians $OG(k, 2n)$ and $OG(k, 2n + 1)$
- Mirrors of $SG$ and $OG$
- Quantum K-theory for symplectic Grassmannians

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Future work

- Isotropic flag varieties, including $G_2$ flags [In progress]
Thank you!
Quantum cohomology ring relations

Consider $SG(n, 2n)$ with $n = 2k + 1$ odd. We can rewrite the chiral ring relations as following:

$$q\sigma_a^{2k} + qe_2(\sigma)\sigma_a^{2k-2} + \cdots + qe_{2k-2}(\sigma)\sigma_a^2 + qe_{2k}(\sigma) = \sigma_a^{4k+2},$$

where $e_i(\sigma)$ is the $i$-th elementary symmetric polynomial.

Since the $e_i(\sigma)$ are Weyl invariant, the $e_i(\sigma)$ are constant on Weyl orbits. Rewrite the above equation as

$$P(\sigma_a^2) \equiv (\sigma_a^2)^{2k+1} - q(\sigma_a^2)^k - \cdots - qe_{2k-2}(\sigma)\sigma_a^2 - qe_{2k}(\sigma) = 0,$$

Then $\{\sigma_a^2\}$ should satisfy relations determined by the coefficients of $P(\sigma_a^2)$ according to Vieta’s formula.
Quantum cohomology ring relations

Vieta’s formula tell us that for the polynomial:

\[ P(x) = (x)^{2k+1} - q(x)^k - \cdots - qe_{2k-2}(\sigma)x^2 - qe_{2k}(\sigma), \]

\[ \equiv (x - x_1)(x - x_2)\cdots(x - x_{2k+1}), \]

then its \( (2k + 1) \) roots \( \{x_a\} \) should satisfy following relations

\[ \sum_{1 \leq a_1 < \cdots < a_\ell \leq n} x_{a_1} \cdots x_{a_\ell} = (-1)^{\ell-1} e_{2\ell-n-1}(\sigma)q, \]

for \( \ell \leq n = 2k + 1 \) and we’ve used \( e_i = 0 \) for \( i < 0 \) and \( e_0 = 1 \).
Quantum cohomology ring relations

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\[ \sum_{1 \leq a_1 < \cdots < a_\ell \leq n} x_{a_1} \cdots x_{a_\ell} = (-)^{\ell-1}e_{2\ell-n-1}(\sigma)q, \]

for \(\ell \leq n = 2k + 1\) and we’ve used \(e_i = 0\) for \(i < 0\) and \(e_0 = 1\).

\[ x_a \rightarrow \sigma_a^2 \quad \Rightarrow \quad \text{It recovers the quantum cohomology ring relations for } SG(n, 2n) \text{ when } k = 2k+1. \]

[Buch, Kresch & Tamvakis, arXiv:0809.4966[math-AG]]

Similar analysis applies to \(SG(n, 2n)\) for \(n\) even cases.
Witten indices for $SG(k, 2n)$ with $k$ odd

For $k$ odd, as discussed before, there are only the geometric phase and the pure Coulomb branch phase. Therefore, the number of Coulomb vacua should match the Euler characteristic of $SG(k, 2n)$:

$$2^k \binom{n}{k}.$$ 

The chiral ring relations,

$$q \prod_{b \neq a} (\sigma_a + \sigma_b) = \sigma_a^{2n},$$

which can also be rewritten as

$$\sigma_a^{2n} - q \left[ \sigma_a^{k-1} + e_2(\sigma)\sigma_a^{k-3} + \cdots + e_{k-1}(\sigma) \right] = 0.$$
Witten indices for $SG(k, 2n)$ with $k$ odd

Since $k$ is odd and we count solutions on the excluded locus

$$\sigma_a \neq 0, \quad \sigma_a \neq \pm \sigma_b \text{ for } a \neq b,$$

there is a $\mathbb{Z}_2$-symmetry among solutions:

If $\{\sigma_1, \ldots, \sigma_k\}$ is one solution, then
$\{-\sigma_1, \ldots, -\sigma_k\}$ is another solution.

Further, all solutions satisfy the same equations due to the Weyl symmetry. So we can count the number of solutions as following:

$$\frac{2n(2n - 2) \cdots (2n - 2k + 2)}{k!} = 2^k \binom{n}{k}.$$
Witten indices for $SG(k, 2n)$ with $k$ even

For $k$ even, the $r \ll 0$ phase is a mixed Higgs-Coulomb branch, so the total Witten index consist of contributions from both Higgs and Coulomb branches. This total number should match the Euler characteristic of $SG(k, 2n)$.

Let us check in the $k = 2$ case. On the Coulomb branch, when $k = 2$, the chiral ring relations are

$$q(\sigma_1 + \sigma_2) = \sigma_1^{2n}, \quad q(\sigma_1 + \sigma_2) = \sigma_2^{2n},$$

which imply $\sigma_1^{2n} = \sigma_2^{2n}$. Exclude $\sigma_1 = \pm \sigma_2$, there are $(2n - 2)$ relations between $\sigma_1$ and $\sigma_2$. Then the chiral ring relation will reduce to an equation of degree $(2n - 1)$, which has $(2n - 1)$ solutions.
Witten indices for $SG(k, 2n)$ with $k$ even

Therefore, the number of Coulomb vacua is

$$\frac{1}{2}(2n - 2)(2n - 1) = (2n - 1)(n - 1).$$

The Higgs branch in $r \ll 0$ is a $SU(2)$ gauge theory with $2n$ fundamentals, which has Witten index $(n - 1)$.

So the total number is

$$(n - 1) + (2n - 1)(n - 1) = 2^2 \binom{n}{2},$$

which matches the Euler characteristic of $SG(2, 2n)$. 