2020.08.19

Grade restriction rule and its applications

Kentaro Hori

Based on

Herbst, Hori, Page 2008

Hori, Romo 2013

Eager, Hori, Knapp, Romo, work in progress

2d (2,2) supersymmetric gauge theory

gauge group G

matter representation V

superpotential $W(\phi)$

twisted superpotential $\widetilde{W}(\sigma)$

2d (2,2) supersymmetric gauge theory

gauge group G compact Lie group

matter representation V representation$/_{\mathbb{C}}$ of G

superpotential $W(\phi)$ G-invariant polynomial of $\phi \in V$

twisted superpotential $\widetilde{W}(\sigma)$

G-invariant polynomial of $\sigma \in \mathfrak{g}_{\mathbb{C}}$

Assume

$U(1)_V$ R-symmetry :

$$\exists\, R_V \in \mathrm{End}(V)^G_{ss} \,, \quad W(\lambda^{R_V}\phi) = \lambda^2 W(\phi)$$

$U(1)_A$ R-symmetry :

- $G \subset SL(V)$ "Calabi-Yau condition"
- $\widetilde{W}(\sigma) = -\langle t, \sigma \rangle$ linear

$$t = \zeta - i\theta \in \mathfrak{g}^{*G}_{\mathbb{C}} = \mathfrak{t}^{*W}_{\mathbb{C}} = \mathfrak{z}^*_{\mathbb{C}}$$

Fayet-Iliopoulos Theta $\in H^2(BG, U(1))$

Classical potential

$$U(\sigma,\phi) = \frac{1}{8e^2}\left|[\sigma,\bar{\sigma}]\right|^2 + \frac{1}{2}|\sigma\phi|^2 + \frac{1}{2}|\bar{\sigma}\phi|^2$$

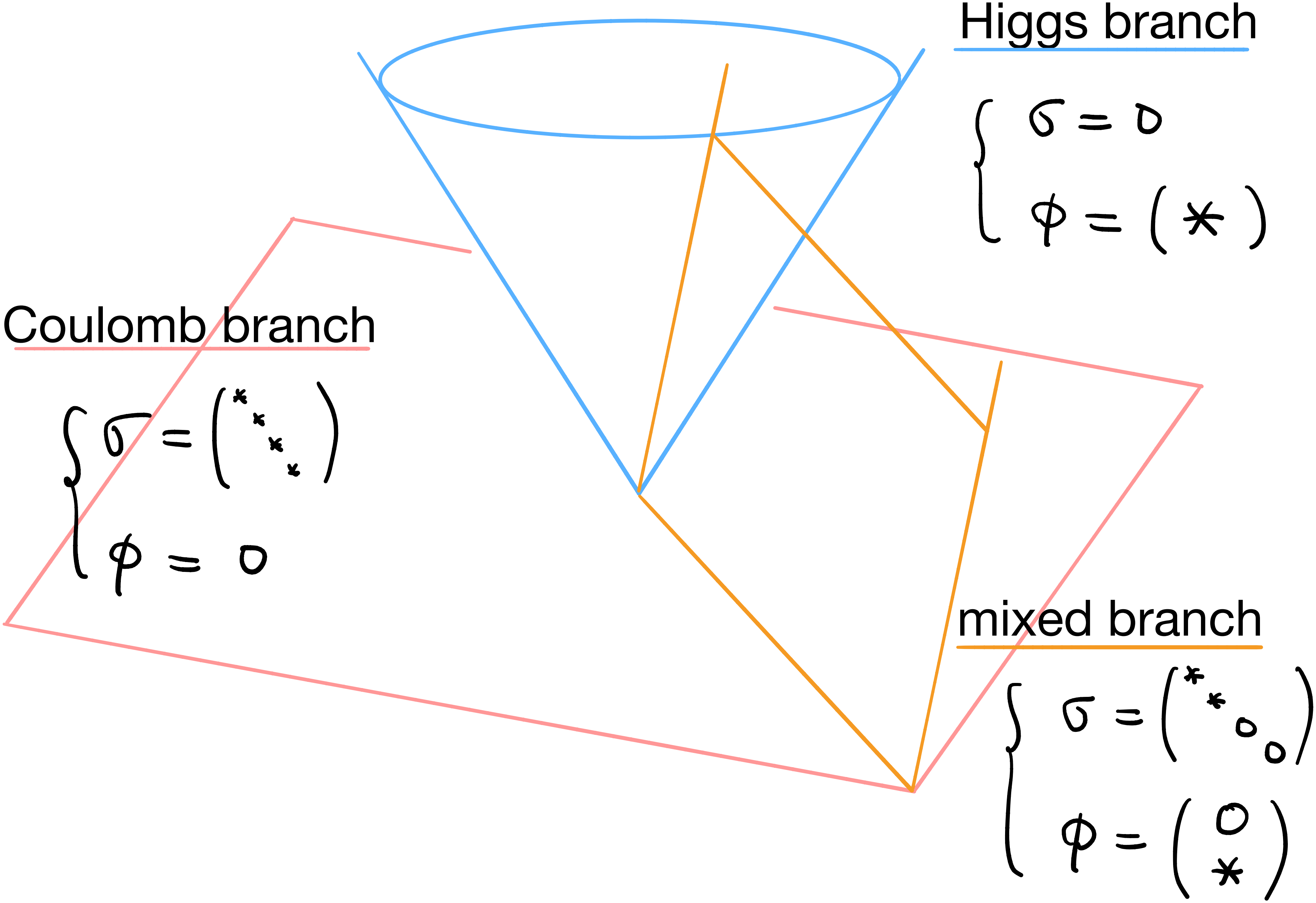
$$+ \frac{e^2}{2}\left(\mu(\phi) - \zeta\right)^2 \quad + \quad |dW(\phi)|^2$$

D-term potential F-term potential

μ : moment map $V \longrightarrow i\mathfrak{g}^*$

$$\langle \mu(\phi), D \rangle := (\phi, D\phi)$$

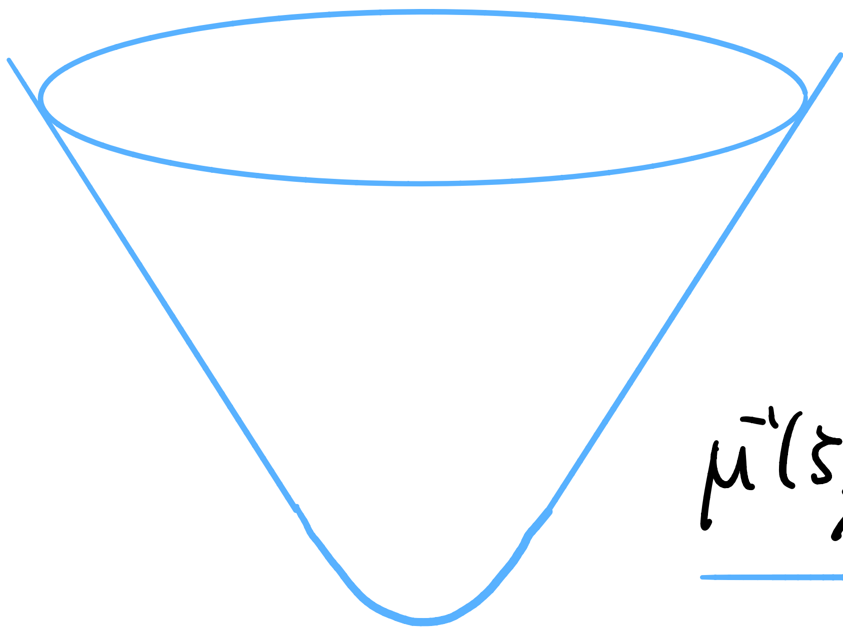
Classical vacua



ζ "generic" : $\mu^{-1}(\zeta) \cap \mathrm{Crit}\, W \circlearrowleft G$

free or finite stabilizers

$\Rightarrow \sigma = 0$



no Coulomb nor mixed

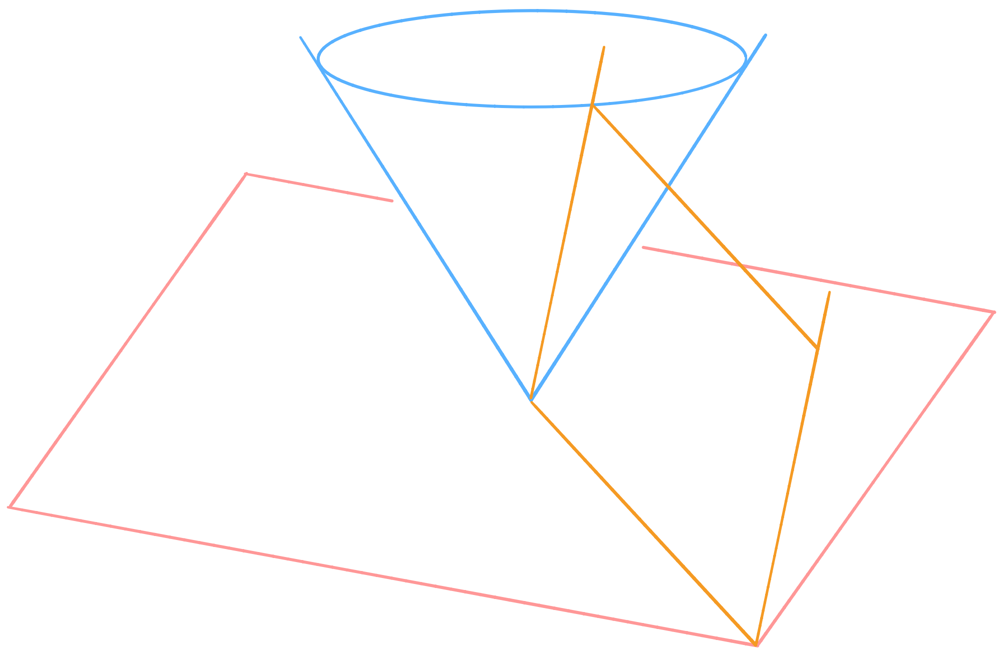
$\mu^{-1}(\zeta)/G$ smooth or orbifold near $\mathrm{Crit}\, W$

$\xrightarrow{\text{RG}}$ LG-model $\left(\mu^{-1}(\zeta)/G,\ W_{\text{ind}}\right) \xrightarrow{\text{RG}}$ σ-model $\left(\mathrm{Crit}\, W_{\text{ind}}\right)$

if W_{ind} Morse-Bott

ζ "non-generic" : $\mu^{-1}(\zeta) \cap \mathrm{Crit}\, W \circlearrowleft G$

$\exists$ points with continuous stabilizers



$\exists$ Coulomb and/or mixed

$\mu^{-1}(\zeta)/G$ badly singular near $\mathrm{Crit}\, W$

However, quantum correction may lift Coulomb and mixed :

$$U_{eff} = \frac{e_{eff}^2}{2} \left| d\widetilde{W}_{eff}(\sigma) \right|^2 > 0 \quad \text{at } |\sigma| \to \infty .$$

We shall only consider "regular theories" :

there is no Coulomb nor mixed branches

after quantum correction is taken into account.

E.g. Supersymmetric QCD

no choice!

$$G = USp(k),\ V = (\mathbb{C}^k)^{\oplus N},\ W \text{ any},\ \widetilde{W} = 0$$

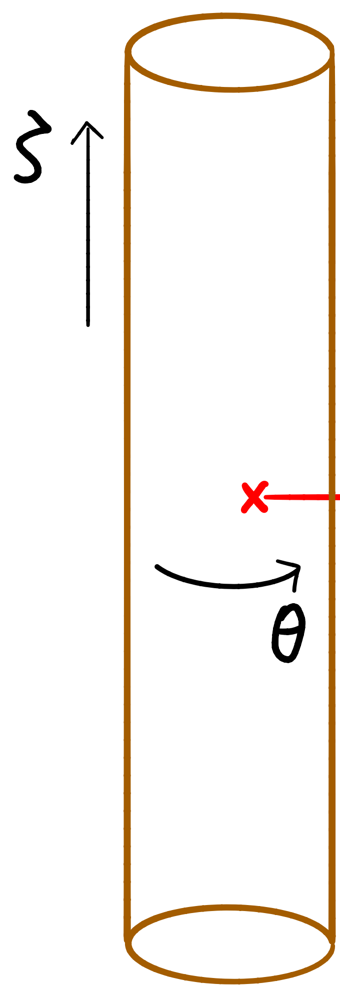
is regular if and only if N is odd.

H 2013

H Tong 2006

E.g. Quintic GLSM Witten 1993

$$G = U(1),\quad V = \underset{p}{\mathbb{C}(-5)} \oplus \underset{x}{\mathbb{C}(1)^5},\quad W = p\,f(x),\quad \widetilde{W} = -t\sigma$$



σ-model $X_f = \{f = 0\} \subset \mathbb{CP}^4$

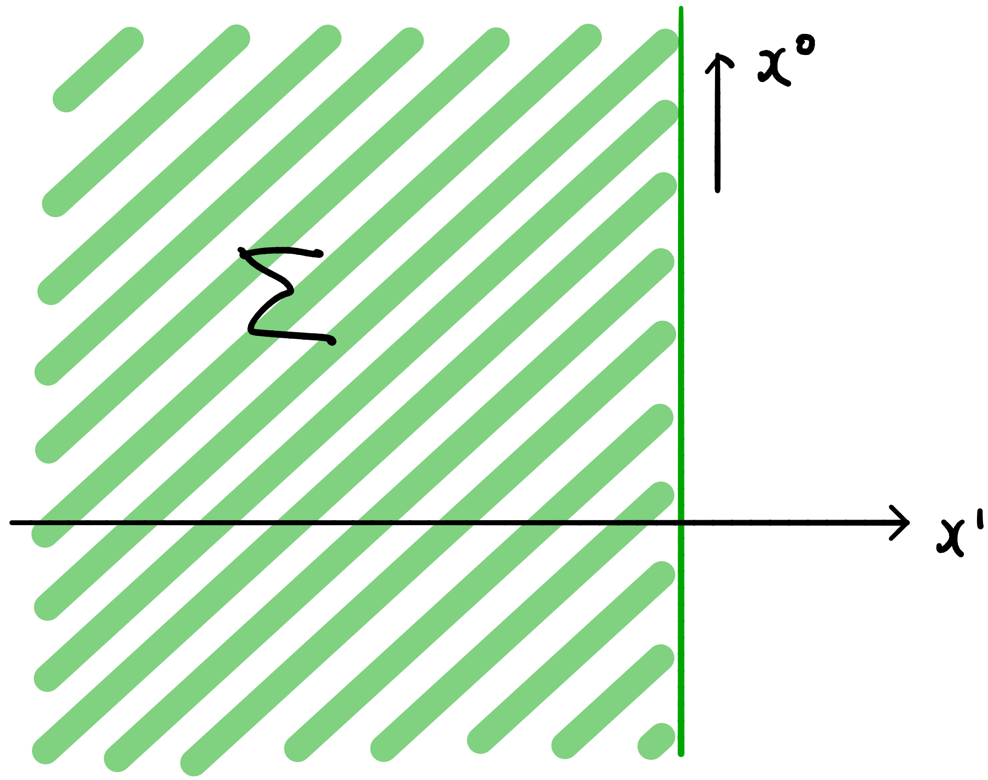
quintic hypersurface

$t \equiv 5\log(-5)$: non-regular

regular everywhere else

LG-orbifold $(\mathbb{C}^5/\mathbb{Z}_5, f)$

Let us formulate the theory on a surface with boundary



and impose boundary conditions preserving

B-type supersymmetry $\quad \overline{Q}_+ + \overline{Q}_-\,,\ \ Q_+ + Q_-\,.$

We shall consider "Neumann" B.C.

$$\begin{cases} D_1\phi - \mathrm{Im}(\sigma)\phi = 0\,, & \cdots \\ v_1 = \mathrm{Im}(\sigma) = 0\,, & \cdots \end{cases}$$

with boundary interaction $P\exp\left(-i\int_{\partial\Sigma} \mathcal{A}_0\, dx^0\right)$

$$\mathcal{A}_0 = [v_0 - \mathrm{Re}\,\sigma]_M$$

$$-\frac{1}{2}\psi^i\partial_i Q(\phi) + \frac{1}{2}\bar{\psi}^{\bar{i}}\partial_{\bar{i}} Q(\phi)^\dagger + \frac{1}{2}\{Q(\phi), Q(\phi)^\dagger\}$$

for Chan-Paton factor M and

matrix factorization $Q(\phi)$ of $W(\phi)$.

- $M = M^{ev} \oplus M^{od}$ $\mathbb{Z}_2$-graded, θ-projective representation of G.

- $Q : V \to \mathrm{End}^{od}(M)$ G-equivariant polynomial function

$$Q(\phi)^2 = W(\phi)\, \mathrm{id}_M .$$

Assume $U(1)_V$ R-symmetry :

$$\exists R_M \in \mathrm{End}^{ev}(M)^G_{ss}, \quad \lambda^{R_M} Q(\lambda^{R_V}\phi)\lambda^{-R_M} = \lambda \cdot Q(\phi).$$

HHP 2008

$\boxed{\text{Question}}$

Does it define a well behaved boundary condition

in the quantum theory ?

$\boxed{\text{Possible failure}}$

The effective boundary potential $V_{eff}(\sigma)$ on the

'Coulomb branch' can be unbounded below.

Regularity may be spoiled in the boundary.

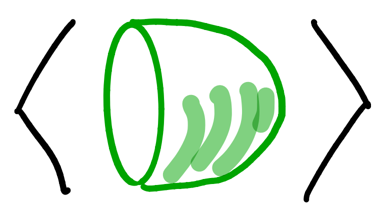
Indeed, for Abelian G, HHP2008 found

$$V_{\mathrm{eff}}(\sigma) = -\mathrm{Re}\langle 2\pi q, \sigma \rangle - \mathrm{Im}\, \widetilde{W}_{\mathrm{eff}}(\sigma)$$

a charge of M

and it can be unbounded below for some q's on the "real locus" $\{\mathrm{Im}\,\sigma = 0\} = i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$.

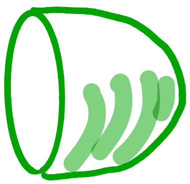
Also, HR2013 found an integral formula for the hemisphere partition function



and it can be divergent for some M's on the real locus.

$T \subset G$ maximal torus,

$V|_{T\times U(1)_V} = \bigoplus_i \mathbb{C}(Q_i, R_i)$ weight decomposition

$\Big\langle$  $\Big\rangle$

$$= \int_\gamma d^l\sigma \prod_{\alpha>0} \langle \alpha, \sigma\rangle \sinh(\pi\langle \alpha, \sigma\rangle) \times \prod_i \Gamma\left(i\langle Q_i, \sigma\rangle + \frac{R_i}{2}\right) \times e^{i\langle t, \sigma\rangle} \cdot \mathrm{tr}_M\left(e^{\pi i R_M} e^{2\pi\sigma}\right)$$

HR, Sugishita-Terashima, Honda-Okuda 2013

It can be divergent for some M's for $\gamma = it$.

A proposal

It defines a well-behaved boundary condition when there is a deformation of the real locus $i\mathfrak{t} \to \gamma$ without hitting the poles $i\langle Q_i, \sigma\rangle + \frac{R_i}{2} = 0, -1, -2, \cdots$ so that the integral is absolutely convergent.

- Deep inside a phase s.t. $G \to$ finite, there is a deformation $it \to \gamma$ that works for any $\mathcal{M}$.

- If $G \to$ continuous,

  (e.g. (regular) theory with simple gauge group near phase boundary of GLSM)

  existence of admissible deformation $it \to \gamma$ imposes a severe constraint on representations of G that can be included in $\mathcal{M}$.

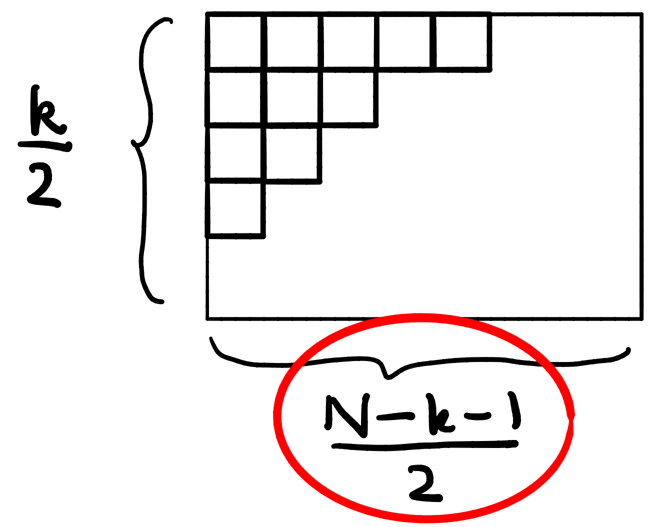
This is the grade restriction rule. HHP2008

Example G simple, V symmetric: $\{\pm Q_i\}_{i>0} \cup \{0's\}$

The representation of highest weight λ can be included in M when $\langle \lambda+\rho, \sigma \rangle < \frac{1}{2} \sum_{i>0} \left| \langle Q_i, \sigma \rangle \right| \quad \forall_{0 \neq} \sigma \in i\mathfrak{t}$.

E.g. super-QCD $G = USp(k)$, $V = (\mathbb{C}^k)^{\oplus N}$ (N odd):

$S_Y(\mathbb{C}^k)^{\otimes |Y|}_0$ for Y in



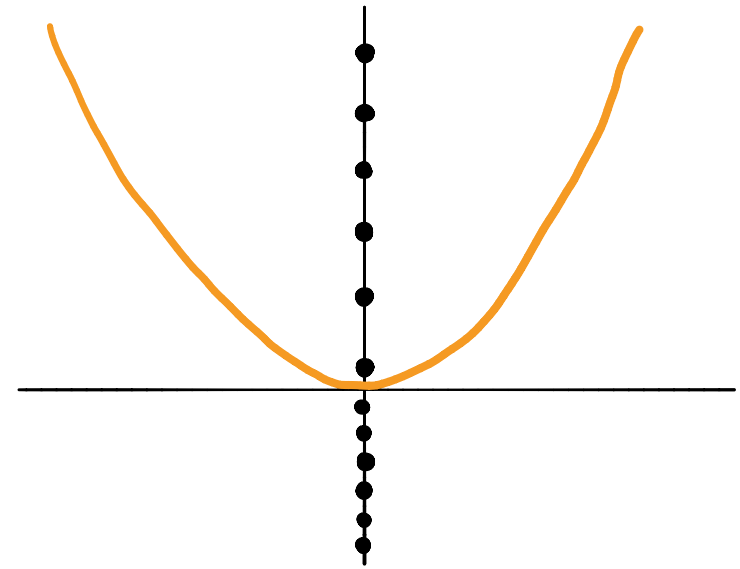
Consistent with 2d Seiberg duality H2011.

Example Quintic GLSM

CY phase $\zeta \gg 0$:

$$\gamma_+ = \{ \operatorname{Im}\sigma = (\operatorname{Re}\sigma)^2 \}$$

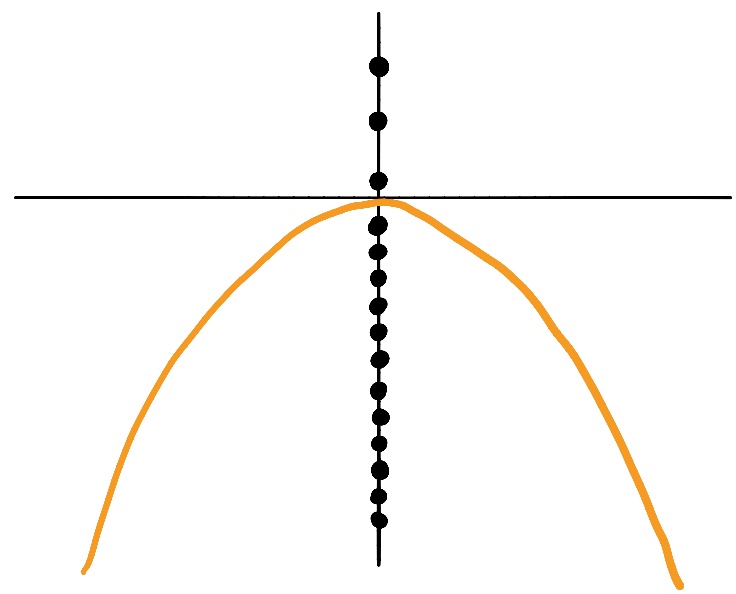
works for any representation.



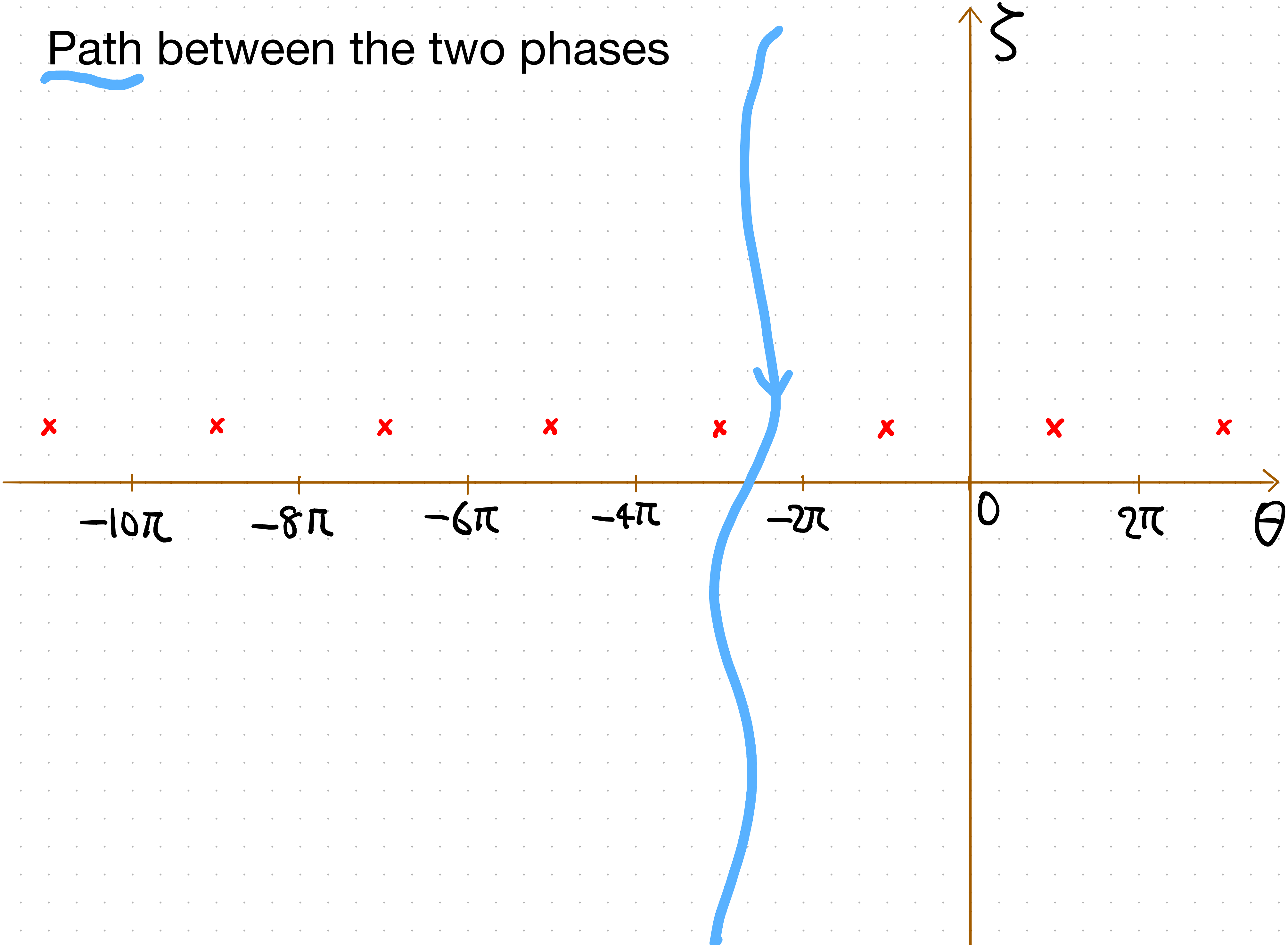
LG phase $\zeta \ll 0$:

$$\gamma_- = \{ \operatorname{Im}\sigma = -(\operatorname{Re}\sigma)^2 \}$$

works for any representation.



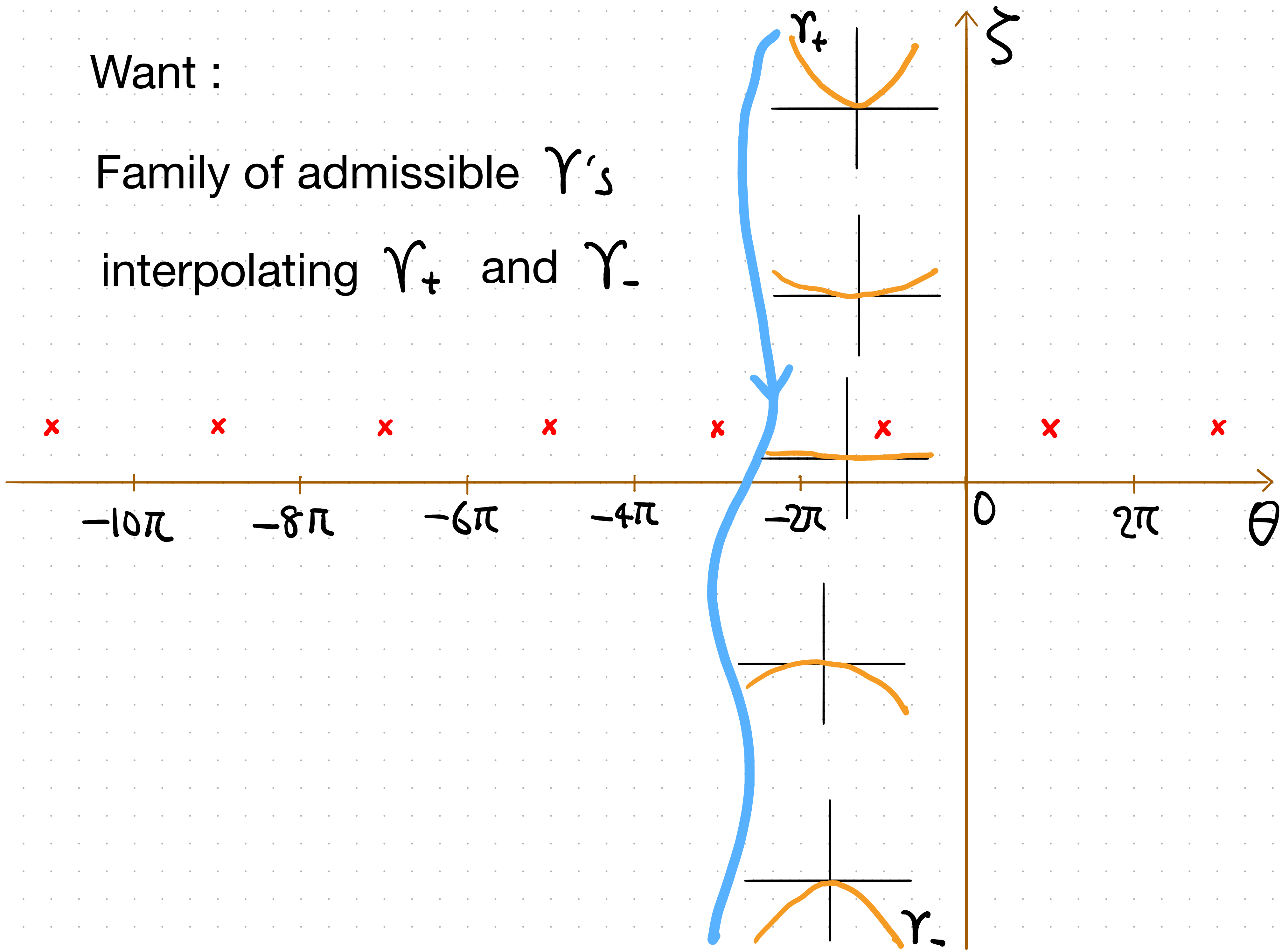
Path between the two phases



Want :

Family of admissible Υ's

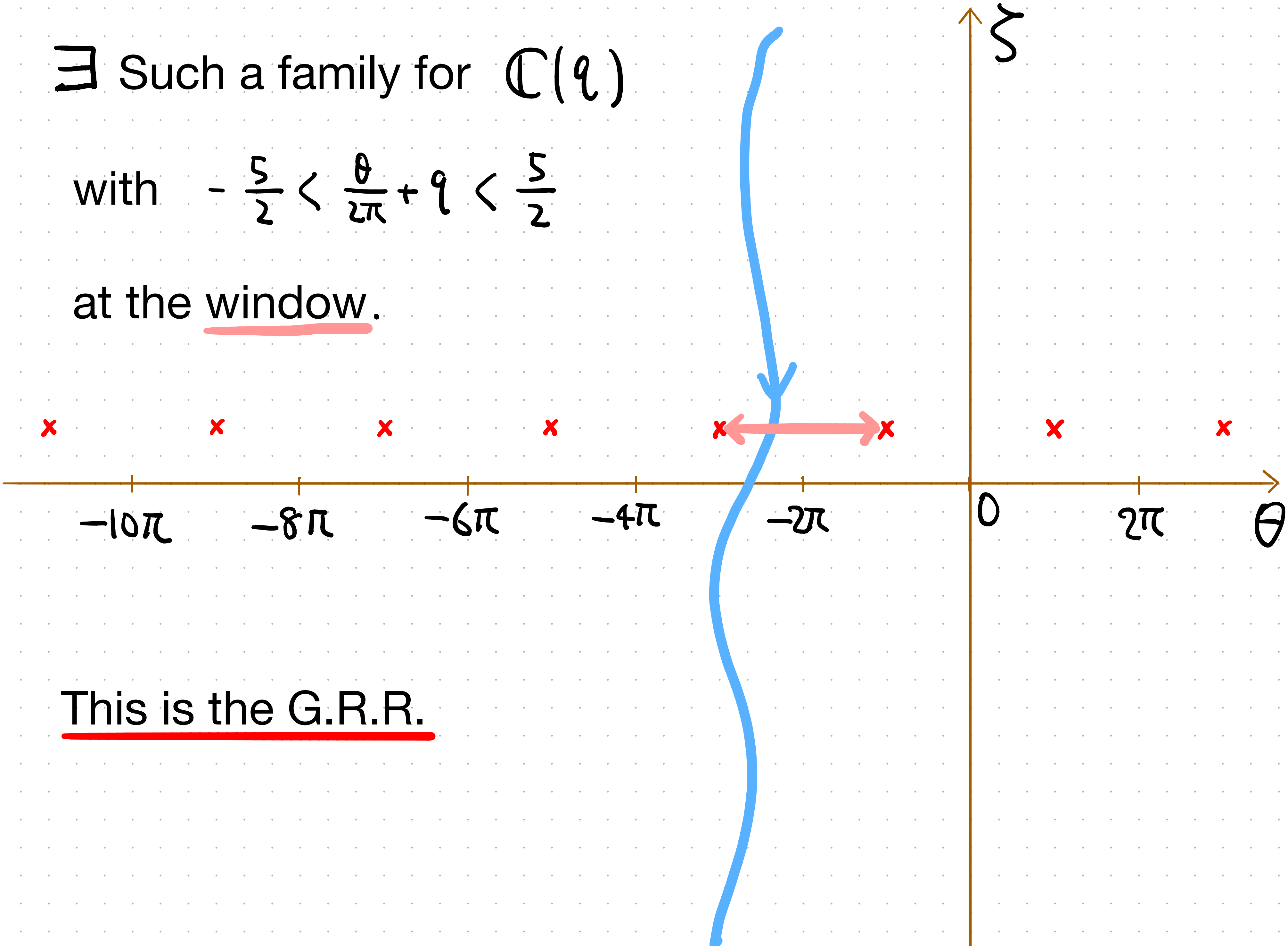
interpolating Υ_+ and Υ_-



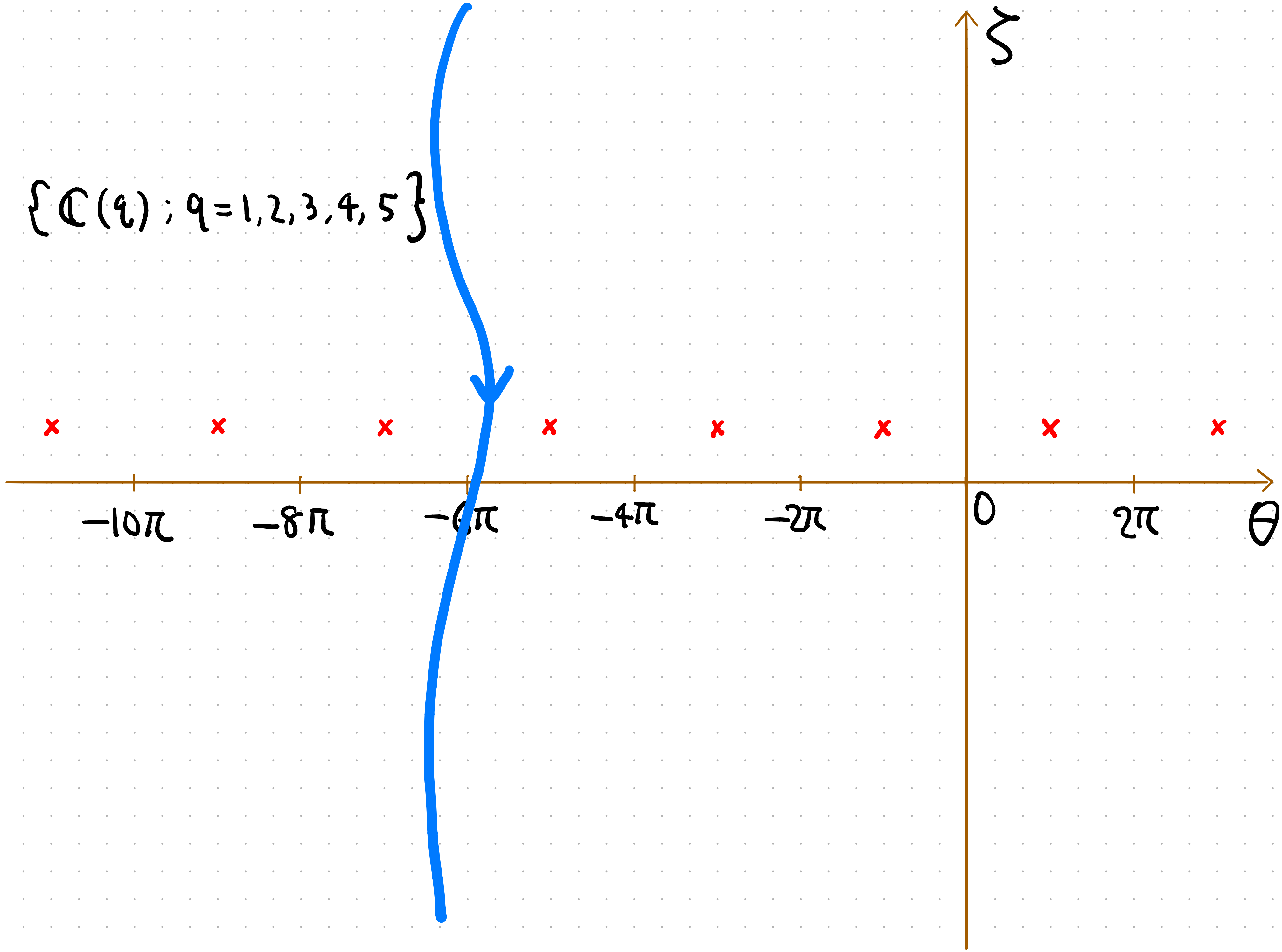
$\exists$ Such a family for $\mathbb{C}(q)$

with $-\frac{5}{2} < \frac{\theta}{2\pi} + q < \frac{5}{2}$

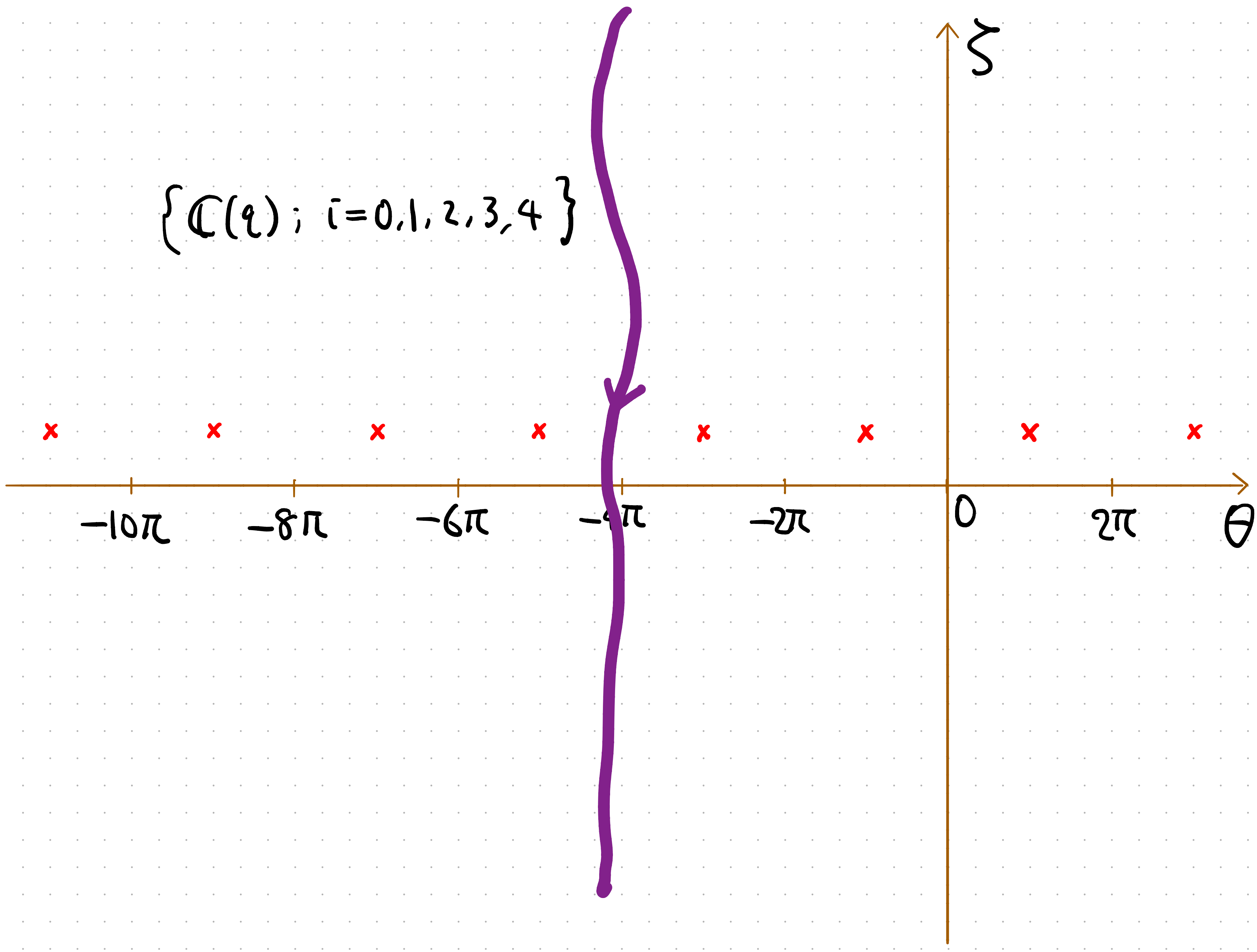
at the window.



This is the G.R.R.



$\{\mathbb{C}(q)\ ;\ q=1,2,3,4,5\}$
ζ
-10π
-8π
-6π
-4π
-2π
0
2π
θ



$\{\mathbb{C}(i);\ i=0,1,2,3,4\}$
ζ
-10π
-8π
-6π
-4π
-2π
0
2π
θ

(M, Q) is grade restricted w.r.t. a window $\mathbf{w}$

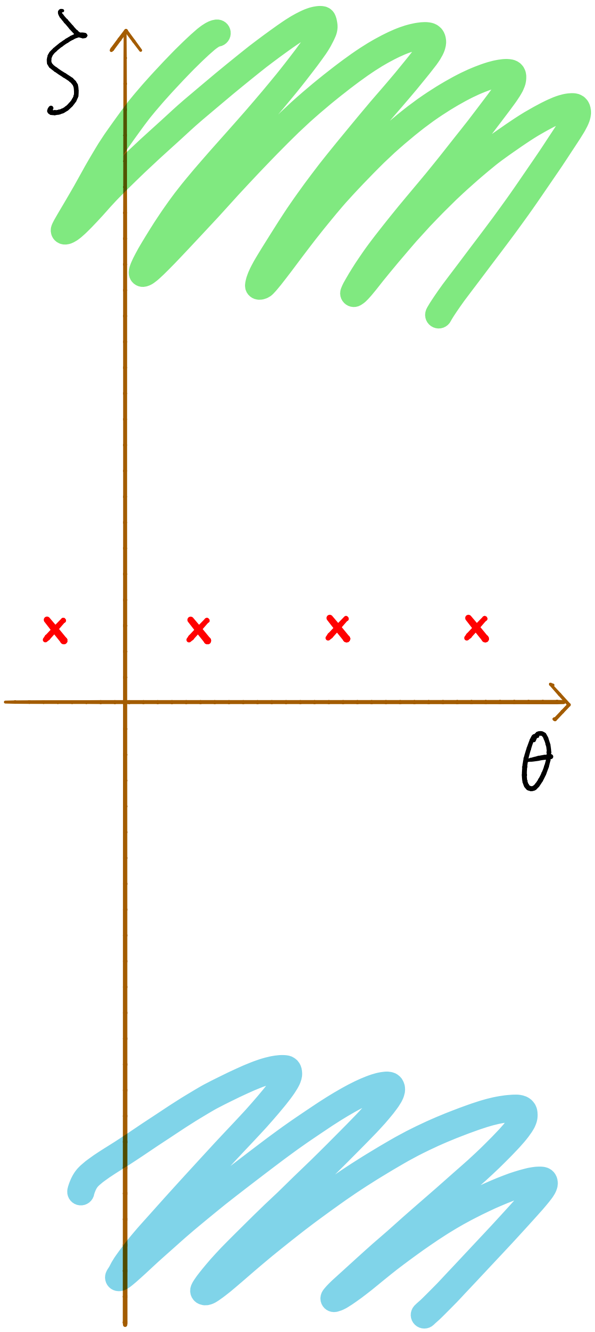
$\Longleftrightarrow$ M is a direct sum of $\mathbb{C}(q)$'s with q

in the range $-\frac{5}{2} < \frac{\theta}{2\pi} + q < \frac{5}{2}$ at $\mathbf{w}$.

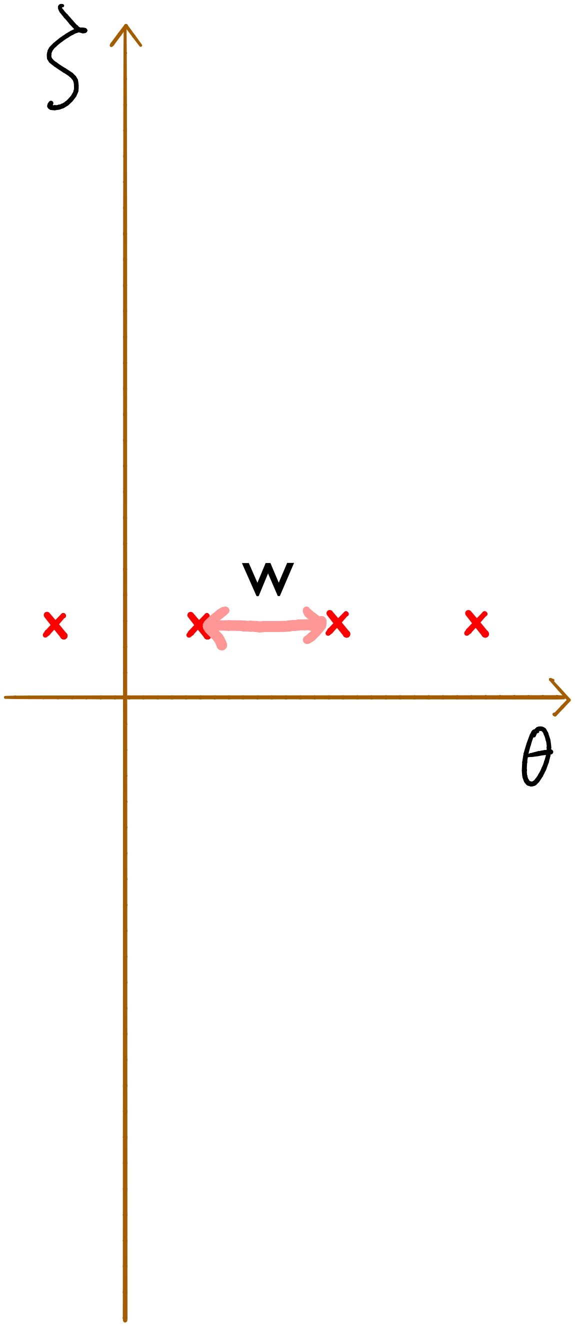
$MF_G(W)$:= the category of G-equivariant matrix factorizations of W

$\cup$

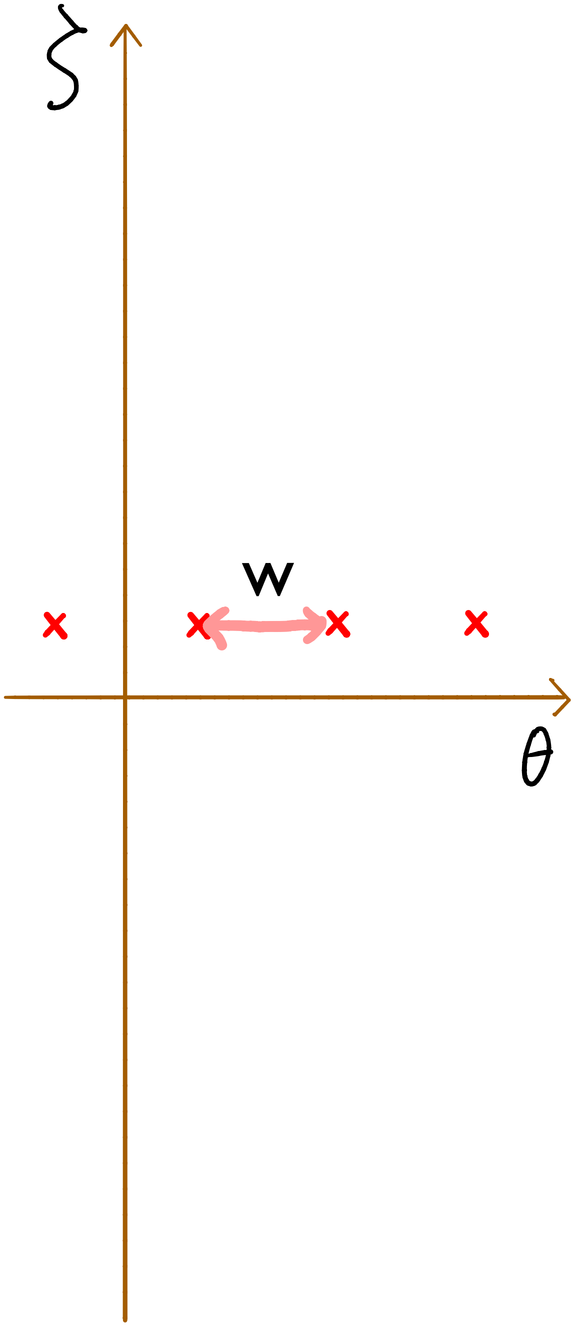
$GR_{\mathbf{w}}$:= the subcategory of grade restricted ones w.r.t. $\mathbf{w}$



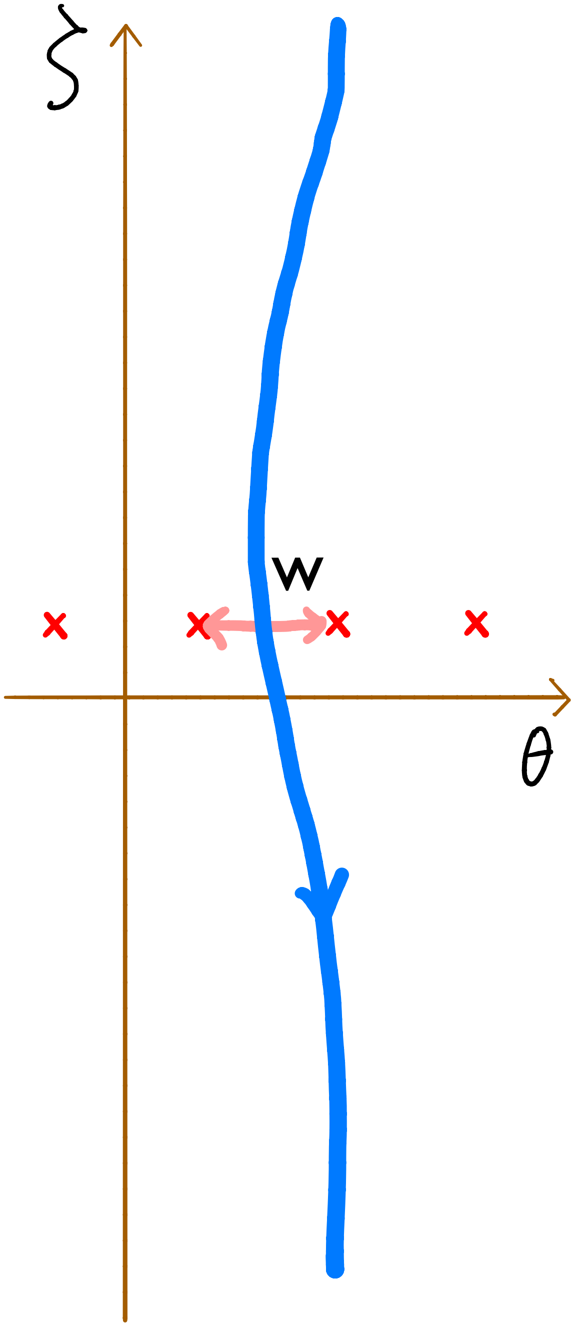
$$\begin{tikzcd}
 & MF_{U(1)}(p \cdot f(x)) \arrow[dl] \arrow[dr] & \\
MF_{\mathbb{Z}_5}(f(x)) & & D^b(X_f)
\end{tikzcd}$$



$$\begin{tikzcd} & MF_{U(1)}(p \cdot f(x)) \arrow[dl] \arrow[dr] \arrow[d, phantom, "\cup \; GR_{w}"] & \\ MF_{\mathbb{Z}_5}(f(x)) & & D^b(X_f) \end{tikzcd}$$

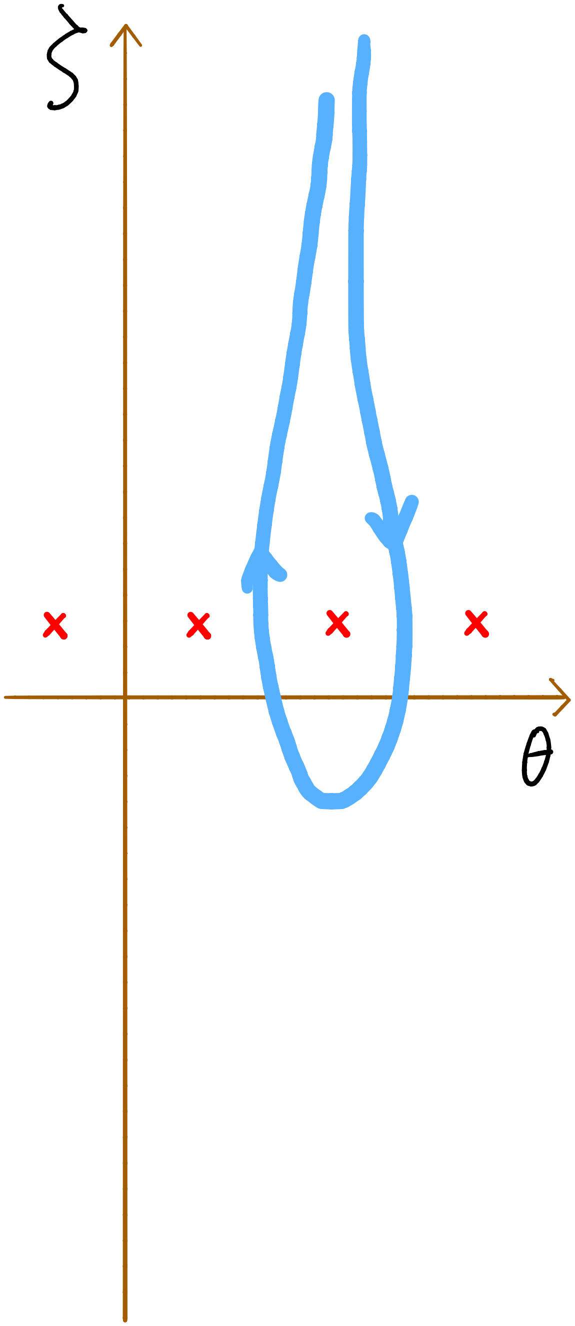


$$
\begin{tikzcd}
 & MF_{U(1)}(p \cdot f(x)) \arrow[ddl] \arrow[ddr] & \\
 & \cup \\
 & GR_{W} \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
MF_{\mathbb{Z}_5}(f(x)) & & D^b(X_f)
\end{tikzcd}
$$



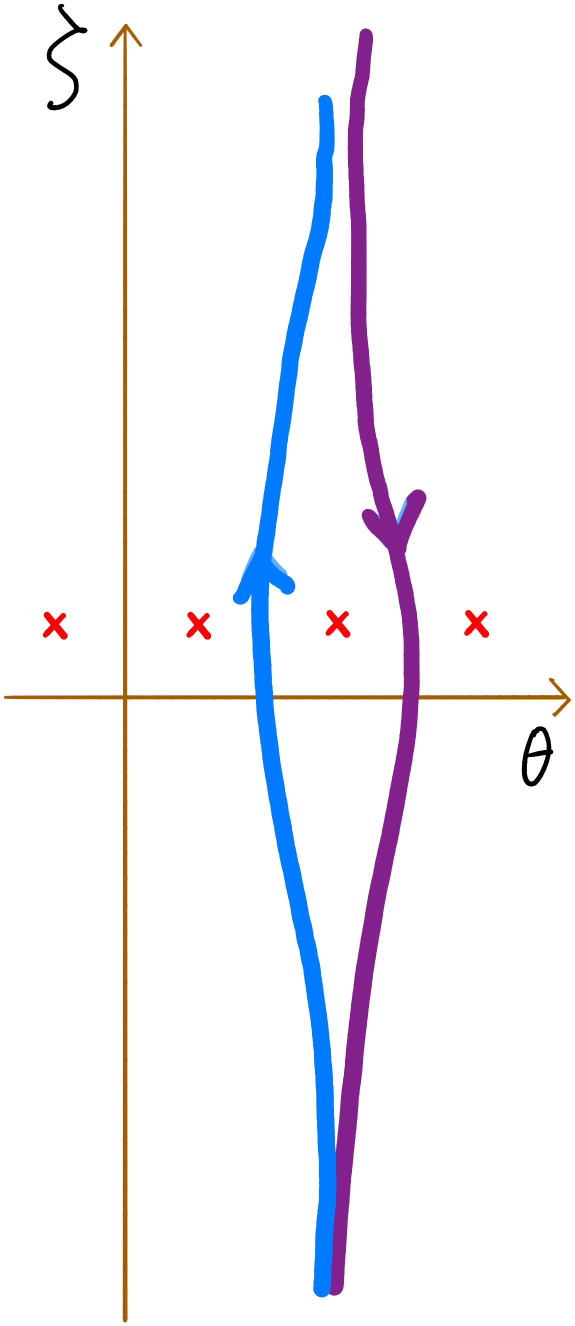
$$
\begin{tikzcd}[column sep=small]
 & \mathrm{MF}_{U(1)}(p\cdot f(x)) \arrow[ldd] \arrow[rdd] & \\
 & \mathrm{GR}_W \arrow[u, phantom, "\cup" description] \arrow[ld, "\cong"'] \arrow[rd, "\cong"] & \\
\mathrm{MF}_{\mathbb{Z}_5}(f(x)) \arrow[rr, phantom, "\cong"] & & D^b(X_f)
\end{tikzcd}
$$

Get an equivalence of categories



What happens when $\tau = \zeta - i\theta$

goes around a singular point ?



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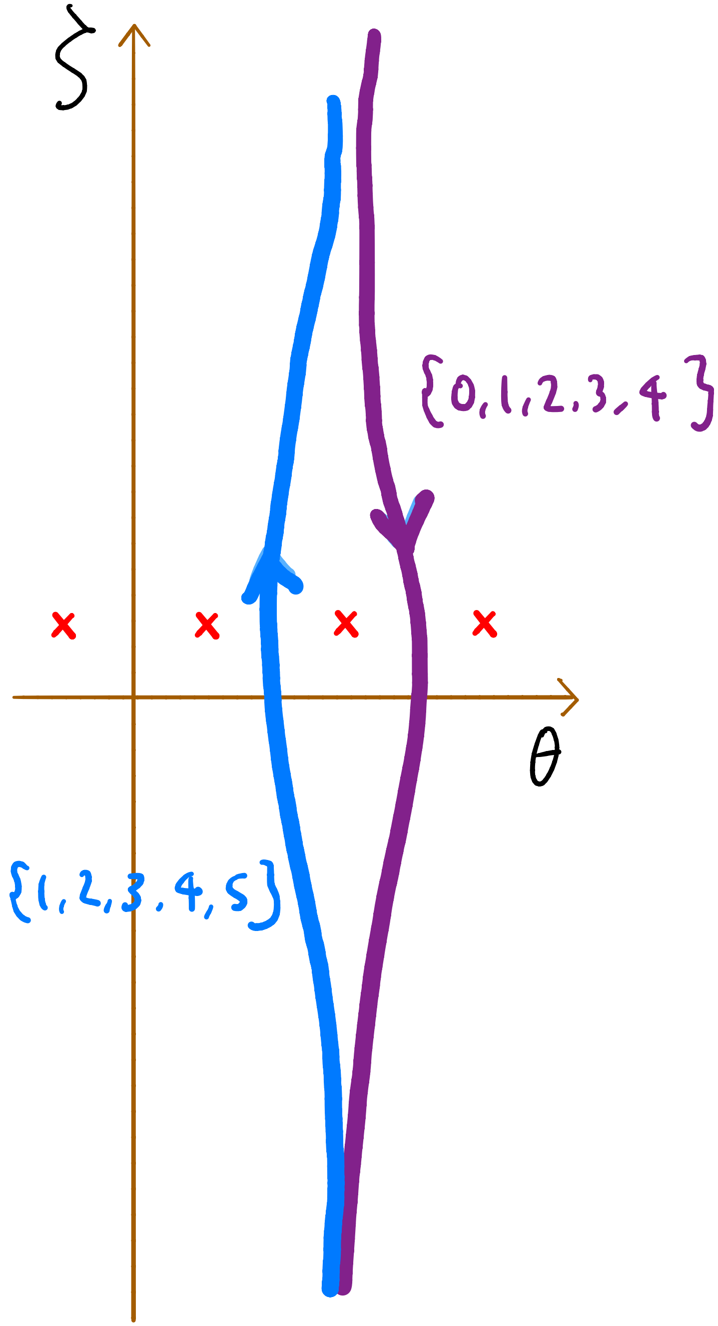
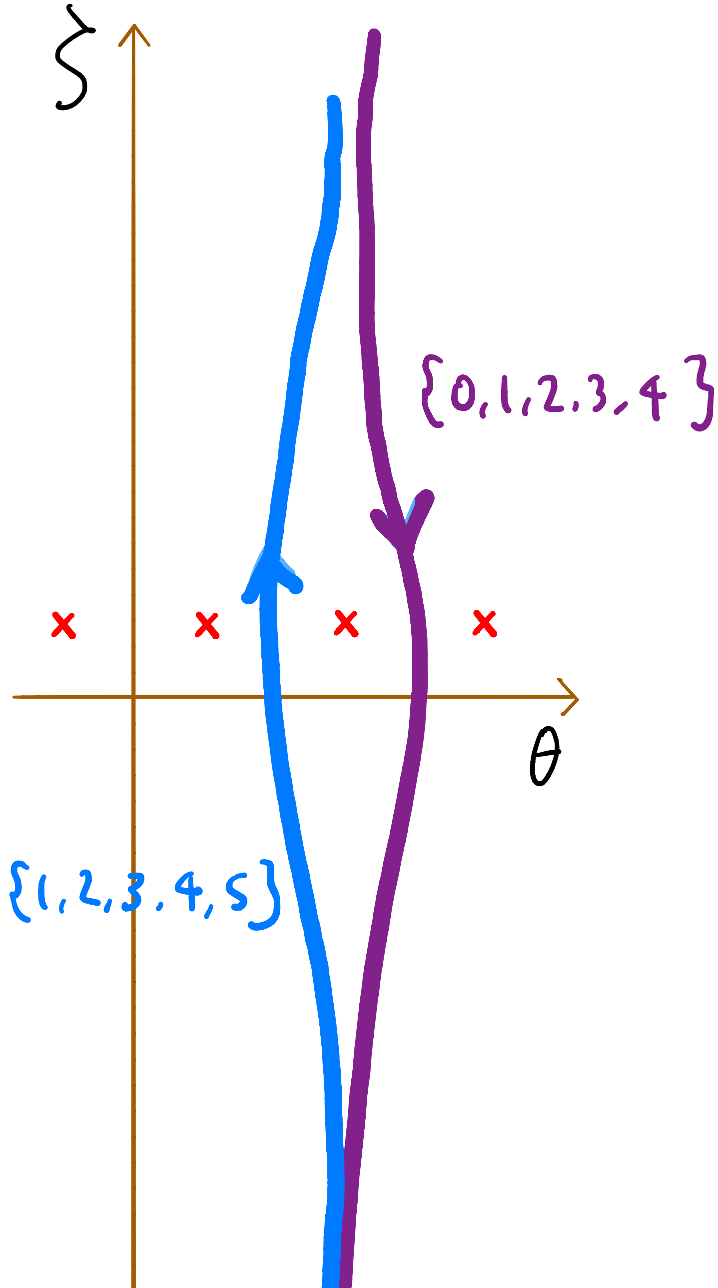
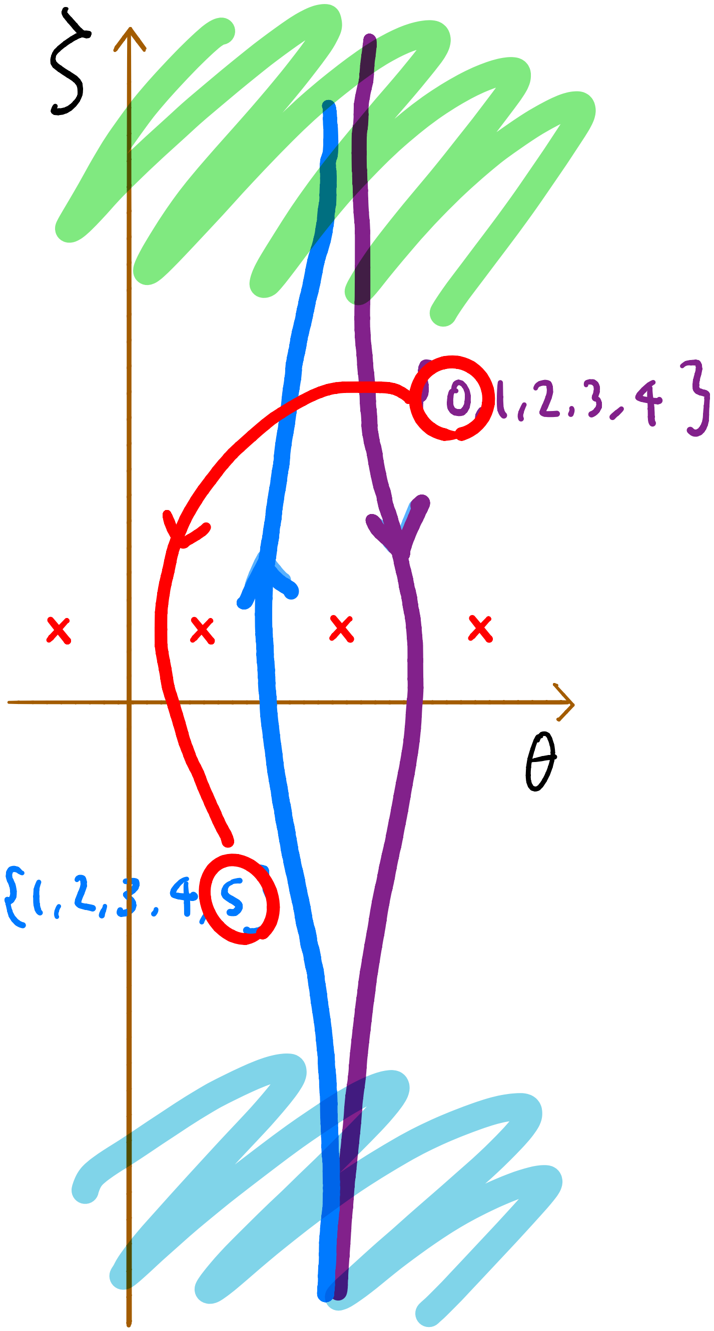


$$
\begin{tikzcd}
 & \mathrm{MF}_{U(1)}(p\cdot f(x)) \arrow[ddl, bend right] \arrow[ddrr, bend left] & & \\
 & \mathrm{GR}_W \arrow[u, phantom, sloped, "\subset"] \arrow[drr] & \mathrm{GR}_{W'} \arrow[ul, phantom, sloped, "\subset"] \arrow[dll] & \\
\mathrm{MF}_{\mathbb{Z}_5}(f(x)) \arrow[ur] & & & D^b(X_f) \arrow[ul]
\end{tikzcd}
$$



$$
\begin{tikzcd}
 & & \mathrm{MF}_{U(1)}(p\cdot f(x)) \arrow[ddll, bend right] \arrow[ddrr, bend left] & & \\
 & \mathrm{GR}_w \arrow[ur, phantom, "\cup"] \arrow[drrr, blue] & & \mathrm{GR}_{w'} \arrow[ul, phantom, "\cup"] \arrow[dlll, purple] & \\
 \mathrm{MF}_{\mathbb{Z}_5}(f(x)) \arrow[ur, blue] & & & & D^b(X_f) \arrow[ul, purple]
\end{tikzcd}
$$

non-trivial autoequivalence



Replacement $GR_{w'} \rightsquigarrow GR_w$

in LG phase can be done with

$$
\begin{tikzcd}
\mathbb{C}(i,0) \arrow[r, shift left, "f(x)"] \arrow[d, mapsto] & \mathbb{C}(i+1,5) \arrow[l, shift left, "p"] \arrow[d, mapsto] \\
0 & \mathcal{O}_{X_f}[i]
\end{tikzcd}
$$

The effect is Seidel-Thomas twist by $\mathcal{O}_{X_f}$:

$$B \mapsto \mathrm{Cone}\left(B \to \bigoplus_i \mathrm{Hom}(B, \mathcal{O}_{X_f}[i]) \otimes \mathcal{O}_{X_f}[i] \right)$$

Example

Rødland GLSM

H Tong 2006

$$G = U(2)$$

$$V = (\det{}^{-1})^{\oplus 7} \oplus (\mathbb{C}^2)^{\oplus 7} \ni (p^1, \cdots, p^7, x_1, \cdots, x_7)$$

$$W = \sum_{i,j,k} A_k^{ij}\, p^k [x_i x_j]$$

$$\widetilde{W} = -t \cdot \mathrm{tr}(\sigma) \qquad t = \zeta - i\theta \in \mathbb{C}/2\pi i \mathbb{Z}$$

<u>Example</u> Rødland GLSM H Tong 2006

$$G = U(2)$$

$$V = (\det{}^{-1})^{\oplus 7} \oplus (\mathbb{C}^2)^{\oplus 7} \Rightarrow (p^1, \cdots, p^7, X_1, \cdots, X_7)$$

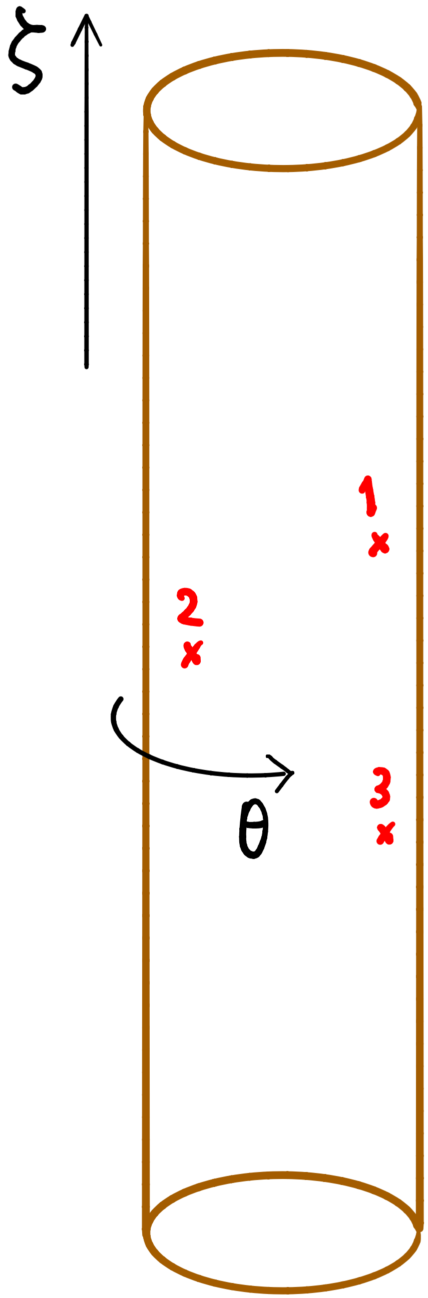
$$W = \sum_{i,j,k} A^{ij}_k \, p^k \, [X_i X_j]$$

$$\widetilde{W} = -t \cdot \mathrm{tr}(\sigma) \qquad t = \zeta - i\theta \in \mathbb{C}/2\pi i \mathbb{Z}$$

$$[X_i X_j] := X_i^1 X_j^2 - X_i^2 X_j^2 \,,$$

write $W = \sum_{ij} A^{ij}(p)[X_i X_j] = \sum_k p^k A_k(X)$

$$\lambda^{R_V}(p, X) = (\lambda^2 p, X)$$



$$X_A = \{ A_1(x) = \cdots = A_7(x) = 0 \} \subset G(2,7)$$

Grassmannian Calabi-Yau

non-regular at $\begin{cases} \zeta_a = 7 \log\left(2\cos\left(\frac{\pi a}{7}\right)\right) \\ \theta_a \equiv \pi a \end{cases}$

$a = \underline{1}, \underline{2}, \underline{3}$

$$Y_A = \{ \mathrm{rk}\, A(p) \leq 4 \} \subset \mathbb{CP}^6$$

Pfaffian Calabi-Yau

The analysis at $\zeta \ll 0$ is non-trivial.

- $p \neq 0$: $U(2) \to SU(2) = USp(2)$
- $USp(2)$ with $(\mathbb{C}^2)^{\oplus 7}$, $W = \sum \underbrace{A^{ij}(p)}_{\text{mass matrix}} [X_i X_j]$ over $\mathbb{CP}^6$
- $\mathrm{rk}\, A(p) = 6$: $USp(2)$ with 1 massless $\mathbb{C}^2$

  $\longrightarrow$ no zero energy state

  $\mathrm{rk}\, A(p) = 4$: $USp(2)$ with 3 massless $\mathbb{C}^2$

  $\longrightarrow$ free theory of $[X_1 X_2]$, $[X_2 X_3]$, $[X_3 X_1]$

$\rightsquigarrow$ σ-model (Y_A).

Let us find the grade restriction rule.

- Parametrize $t_{\mathbb{C}} \ni \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$

- Representations of $U(2)$:

$$S^{\ell}(i) = \mathrm{Sym}^{\ell}\,\mathbb{C}^2 \otimes \det{}^{\otimes i}$$

$$\ell = 0, 1, 2, 3, \cdots; \quad i \in \mathbb{Z}$$

Grassmannian phase $\zeta \gg 0$:

$$\gamma_+ = \left\{ \begin{array}{c} \mathrm{Im}\,\sigma_a = (\mathrm{Re}\,\sigma_a)^2 \\ a=1,2 \end{array} \right\}$$ works for $\forall\, S^\ell(i)$

$\ell = 0, 1, \cdots, \ i \in \mathbb{Z}.$

Pfaffian phase $\zeta \ll 0$:

$$\gamma_- = \left\{ \begin{array}{l} \mathrm{Im}\,\sigma_1 = \mathrm{Im}\,\sigma_2 \\ = -(\mathrm{Re}\,\sigma_1 + \mathrm{Re}\,\sigma_2)^2 \end{array} \right\}$$ is best possible.

It works for $\mathbb{C}(i)$, $S(i)$, $S^2(i)$, $i \in \mathbb{Z}.$

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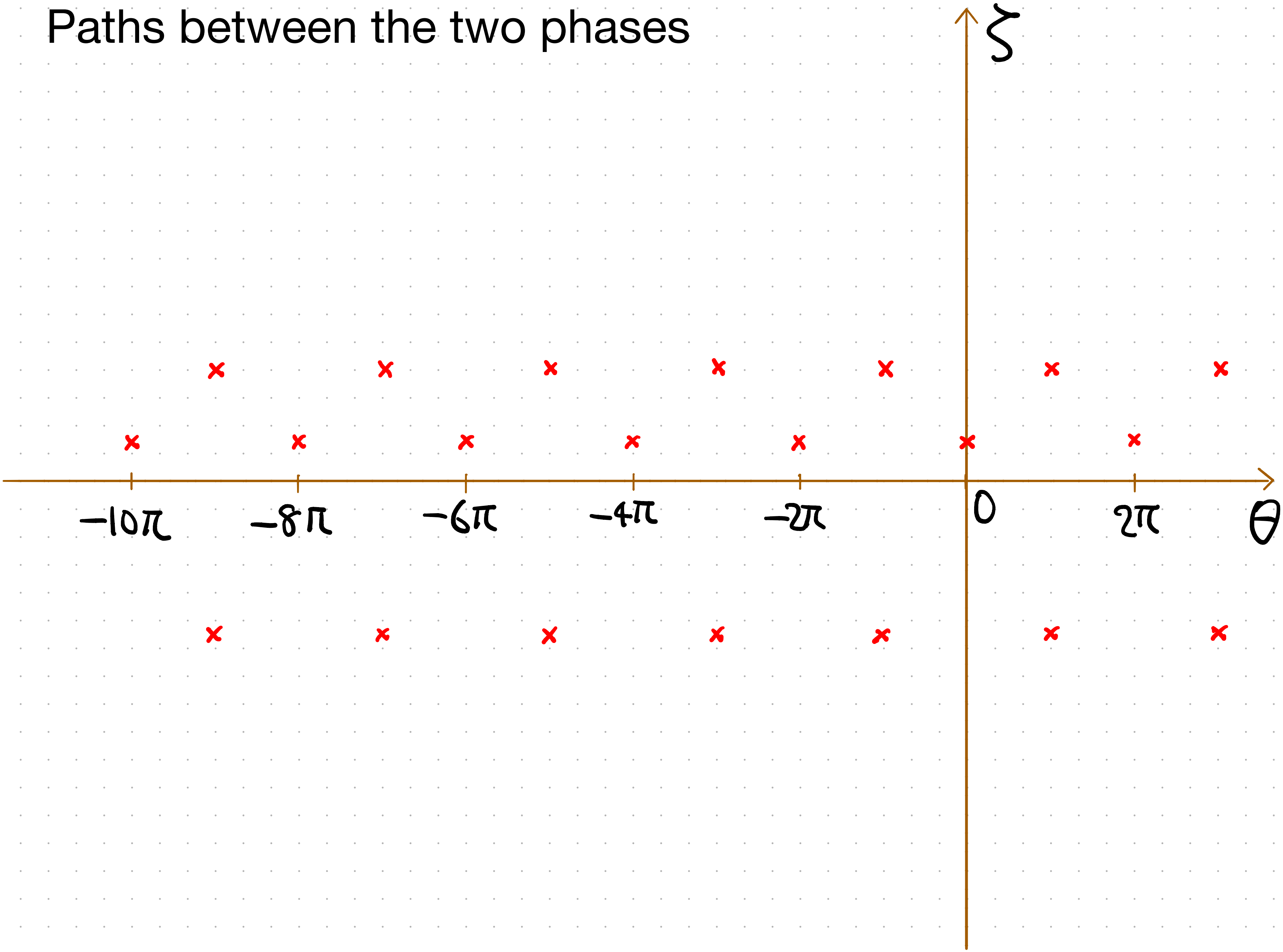
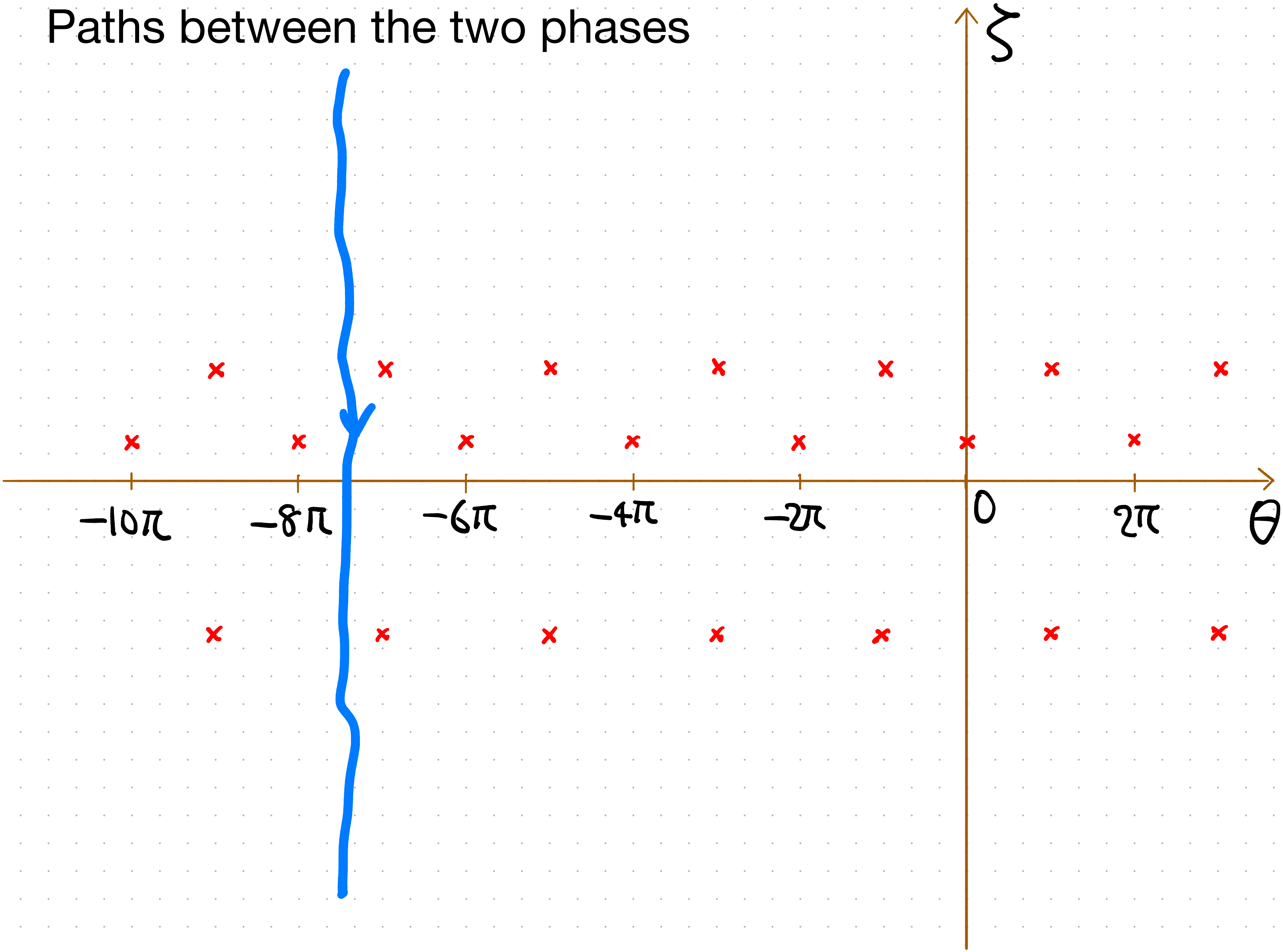
It works for $\mathbb{C}(i), \; S(i), \; S^2(i), \quad i \in \mathbb{Z}.$

$$\underbrace{\bullet \qquad \square \qquad \square\square}_{\text{GRR for } USp(2) \text{ with } (\mathbb{C}^2)^{\oplus 7}}$$

Paths between the two phases



Paths between the two phases



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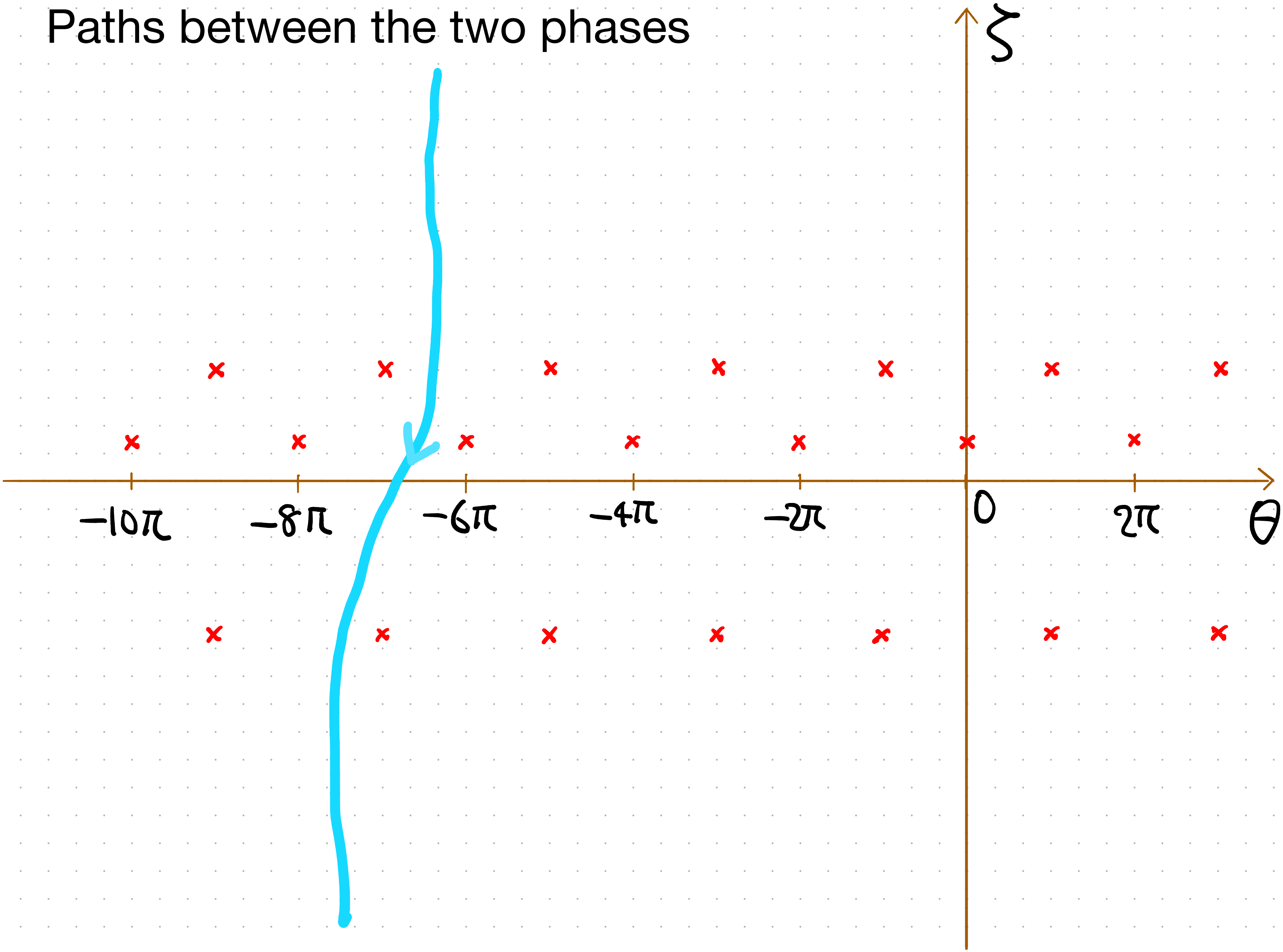
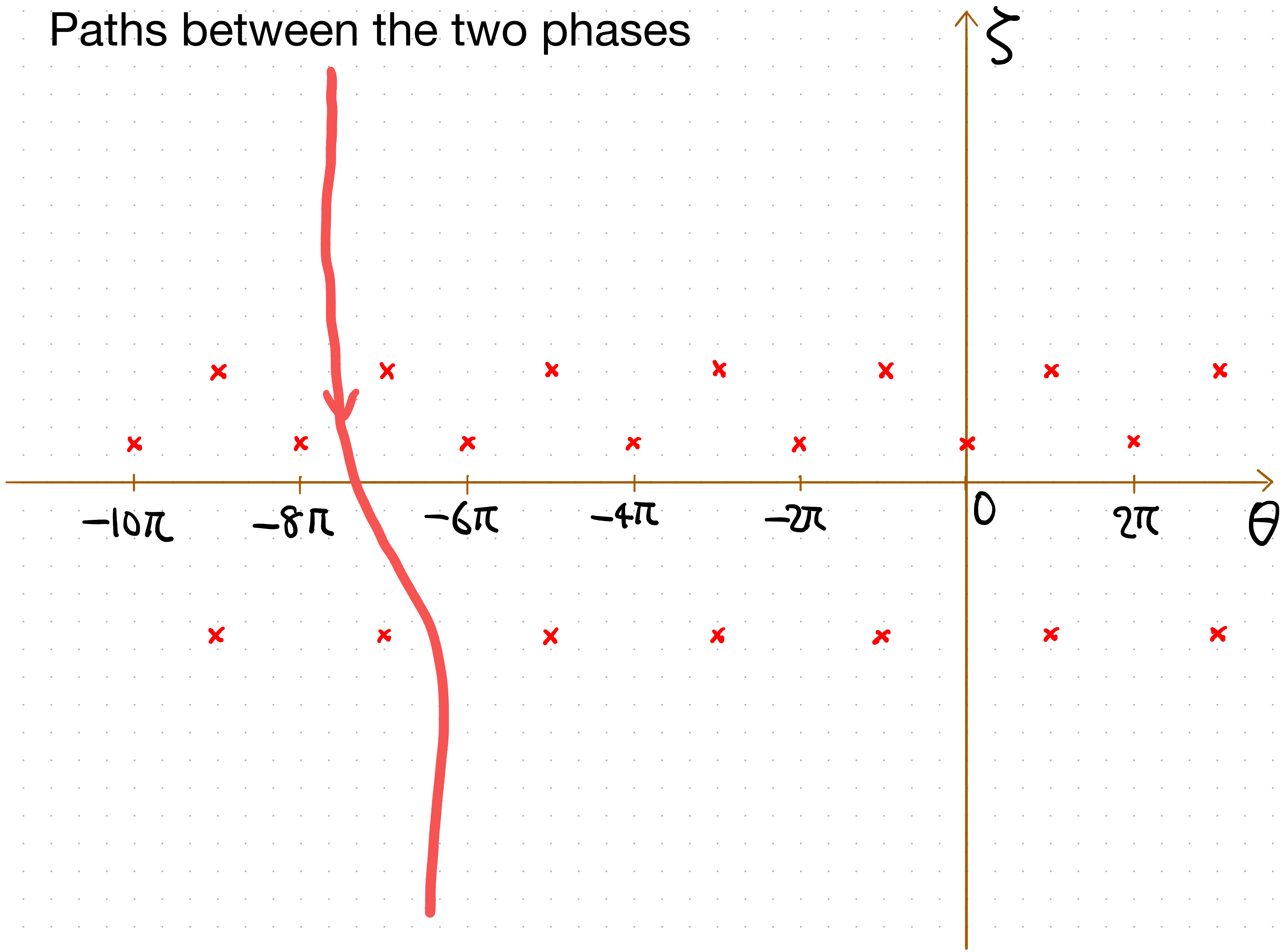
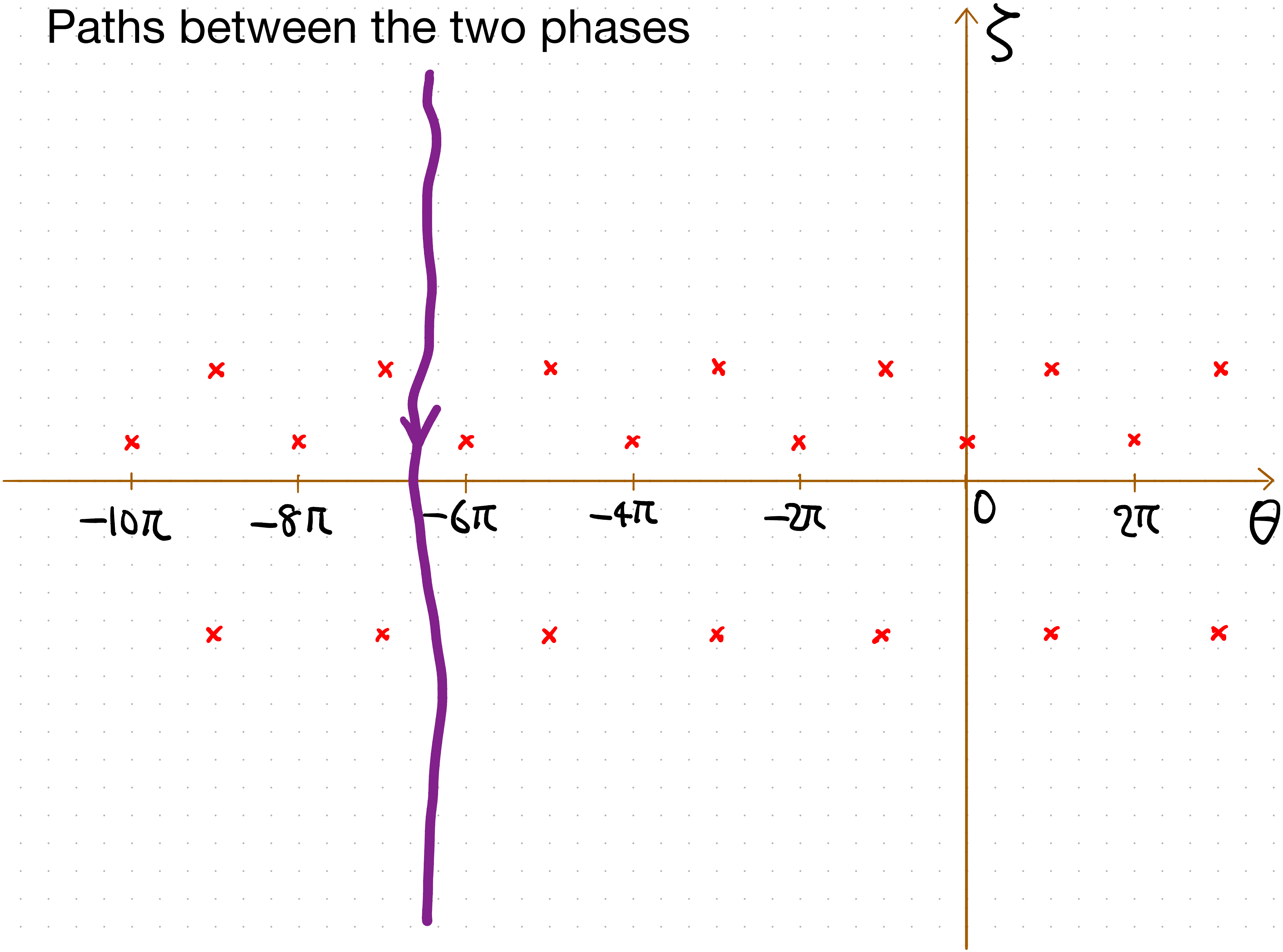


Paths between the two phases

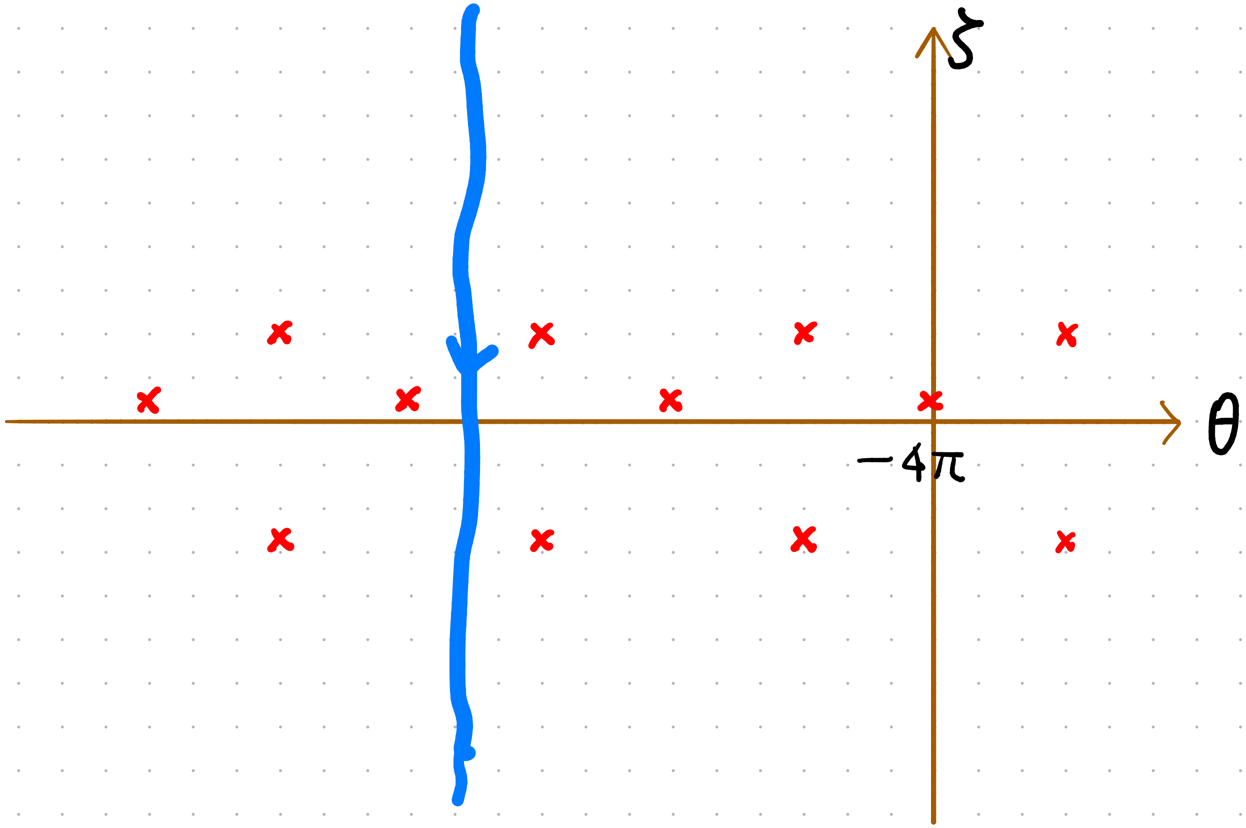


Paths between the two phases

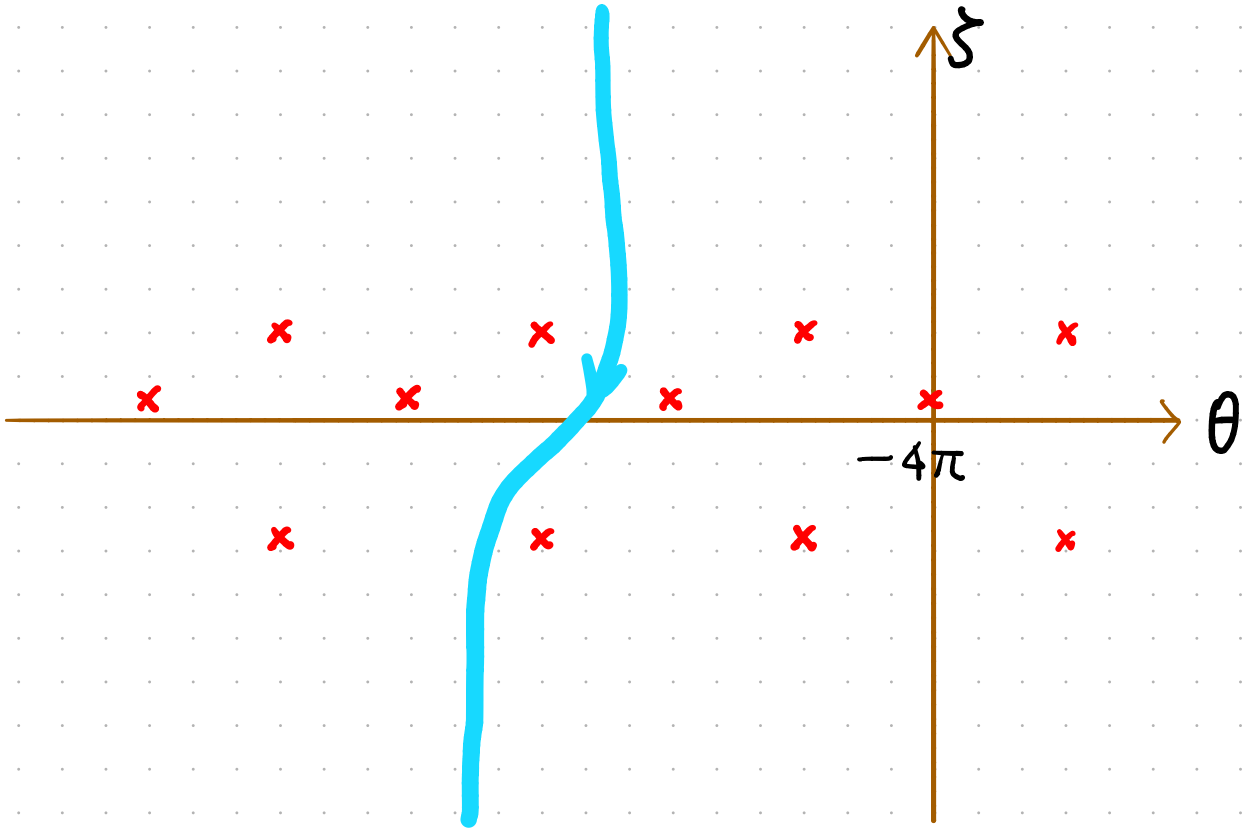


We may only consider $\{\mathbb{C}(i),\ \mathcal{S}(i),\ \mathcal{S}^2(i)\}_{i\in\mathbb{Z}}$

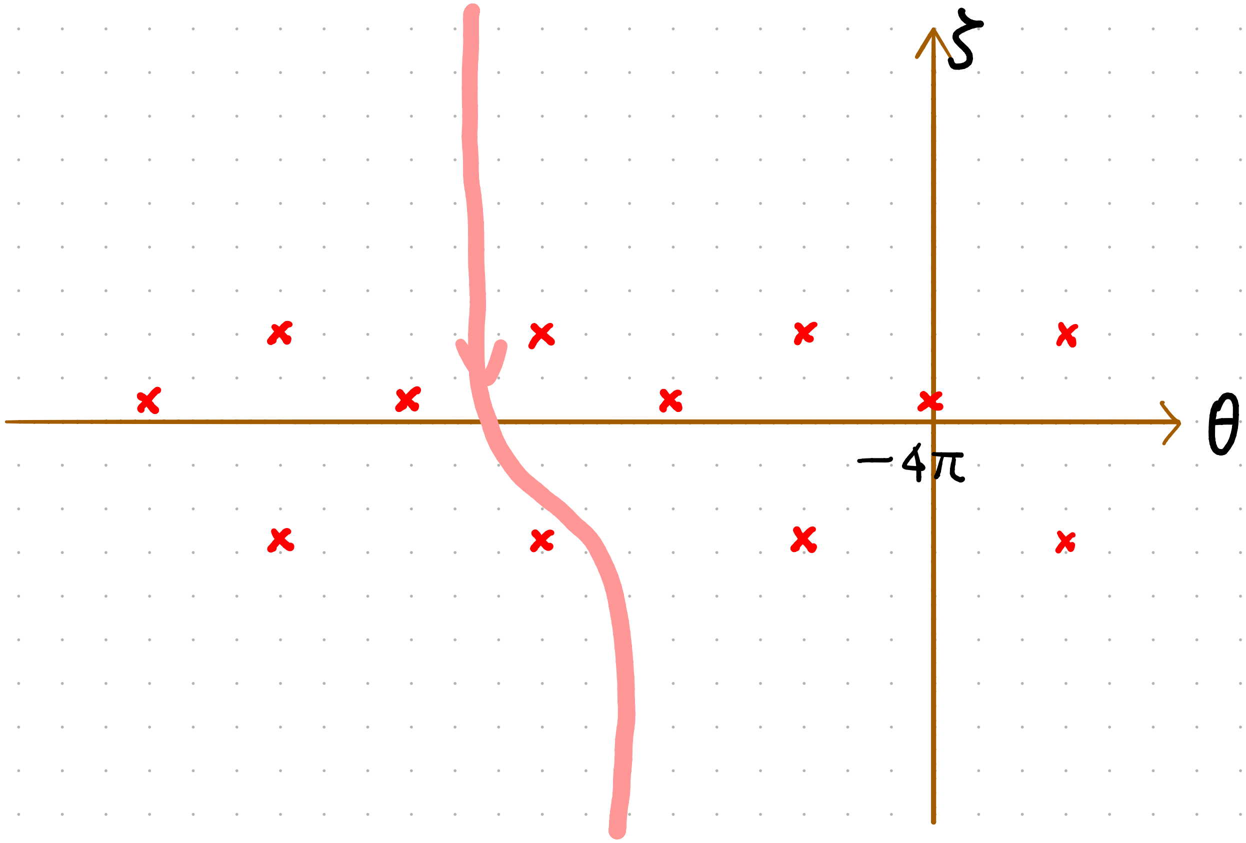
$$\begin{array}{ccccccccccccc}
\cdots & \mathbb{C}(-2) & \mathbb{C}(-1) & \mathbb{C} & \mathbb{C}(1) & \mathbb{C}(2) & \mathbb{C}(3) & \mathbb{C}(4) & \mathbb{C}(5) & \mathbb{C}(6) & \mathbb{C}(7) & \mathbb{C}(8) & \cdots \\
\cdots & \mathcal{S}(-2) & \mathcal{S}(-1) & \mathcal{S} & \mathcal{S}(1) & \mathcal{S}(2) & \mathcal{S}(3) & \mathcal{S}(4) & \mathcal{S}(5) & \mathcal{S}(6) & \mathcal{S}(7) & \mathcal{S}(8) & \cdots \\
\cdots & \mathcal{S}^2(-2) & \mathcal{S}^2(-1) & \mathcal{S}^2 & \mathcal{S}^2(1) & \mathcal{S}^2(2) & \mathcal{S}^2(3) & \mathcal{S}^2(4) & \mathcal{S}^2(5) & \mathcal{S}^2(6) & \mathcal{S}^2(7) & \mathcal{S}^2(8) & \cdots
\end{array}$$



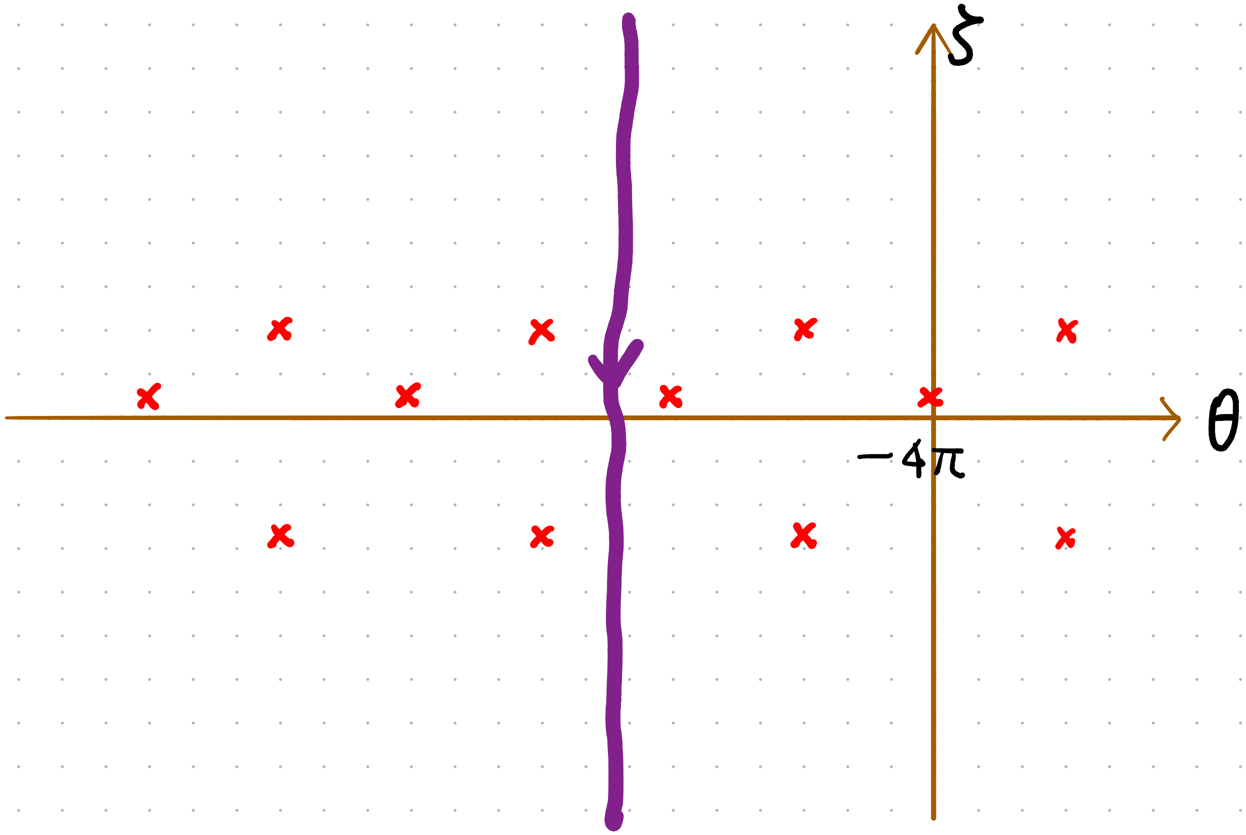
$$\begin{array}{ccccccccccccc}
\cdots & \mathbb{C}(-2) & \mathbb{C}(-1) & \mathbb{C} & \mathbb{C}(1) & \mathbb{C}(2) & \mathbb{C}(3) & \mathbb{C}(4) & \mathbb{C}(5) & \mathbb{C}(6) & \mathbb{C}(7) & \mathbb{C}(8) & \cdots \\
\cdots & \mathbb{S}(-2) & \mathbb{S}(-1) & \mathbb{S} & \mathbb{S}(1) & \mathbb{S}(2) & \mathbb{S}(3) & \mathbb{S}(4) & \mathbb{S}(5) & \mathbb{S}(6) & \mathbb{S}(7) & \mathbb{S}(8) & \cdots \\
\cdots & \mathbb{S}^2(-2) & \mathbb{S}^2(-1) & \mathbb{S}^2 & \mathbb{S}^2(1) & \mathbb{S}^2(2) & \mathbb{S}^2(3) & \mathbb{S}^2(4) & \mathbb{S}^2(5) & \mathbb{S}^2(6) & \mathbb{S}^2(7) & \mathbb{S}^2(8) & \cdots
\end{array}$$



$\cdots$	$\mathbb{C}(-2)$	$\mathbb{C}(-1)$	$\mathbb{C}$	$\mathbb{C}(1)$	$\mathbb{C}(2)$	$\mathbb{C}(3)$	$\mathbb{C}(4)$	$\mathbb{C}(5)$	$\mathbb{C}(6)$	$\mathbb{C}(7)$	$\mathbb{C}(8)$	$\cdots$
$\cdots$	$S(-2)$	$S(-1)$	S	$S(1)$	$S(2)$	$S(3)$	$S(4)$	$S(5)$	$S(6)$	$S(7)$	$S(8)$	$\cdots$
$\cdots$	$S^2(-2)$	$S^2(-1)$	S^2	$S^2(1)$	$S^2(2)$	$S^2(3)$	$S^2(4)$	$S^2(5)$	$S^2(6)$	$S^2(7)$	$S^2(8)$	$\cdots$



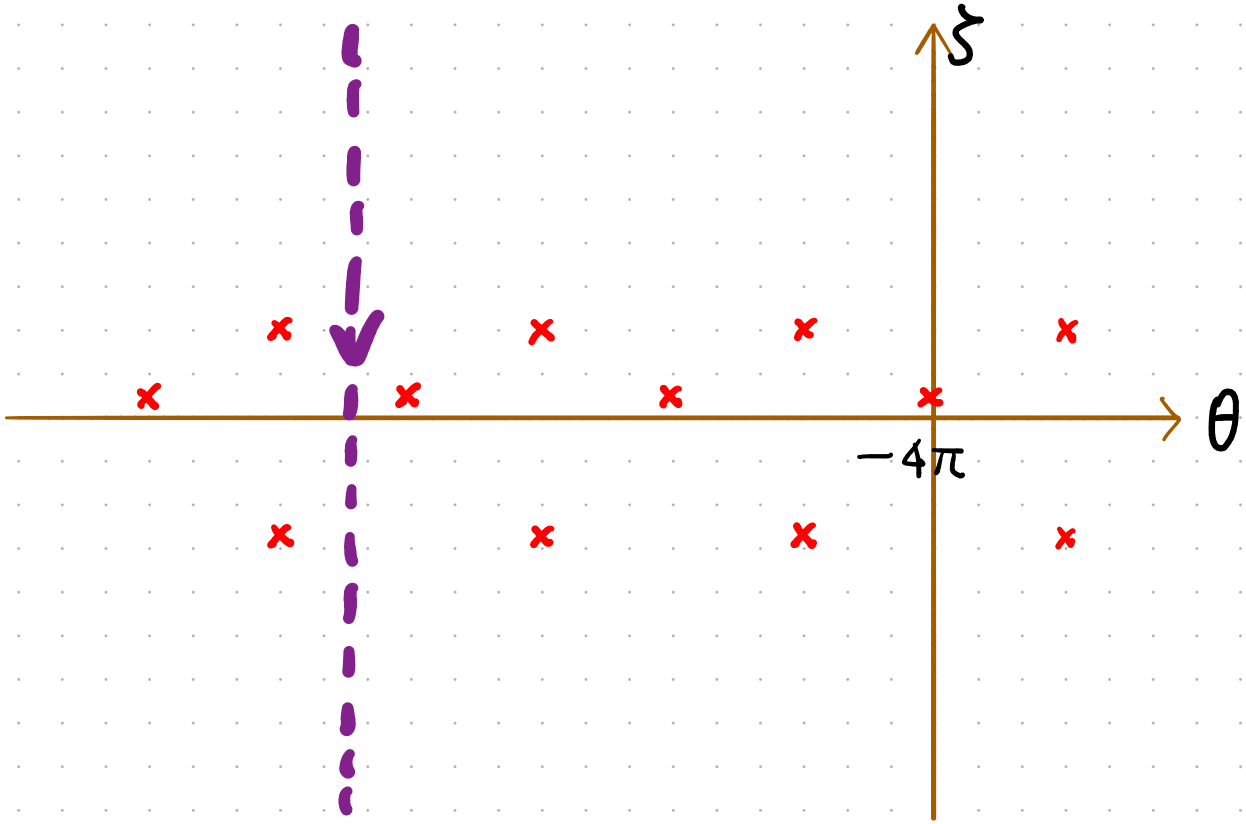
$\cdots$ $\mathbb{C}(-2)$	$\mathbb{C}(-1)$	$\mathbb{C}$	$\mathbb{C}(1)$	$\mathbb{C}(2)$	$\mathbb{C}(3)$	$\mathbb{C}(4)$	$\mathbb{C}(5)$	$\mathbb{C}(6)$	$\mathbb{C}(7)$	$\mathbb{C}(8)$ $\cdots$
$\cdots$ $\mathbb{S}(-2)$	$\mathbb{S}(-1)$	$\mathbb{S}$	$\mathbb{S}(1)$	$\mathbb{S}(2)$	$\mathbb{S}(3)$	$\mathbb{S}(4)$	$\mathbb{S}(5)$	$\mathbb{S}(6)$	$\mathbb{S}(7)$	$\mathbb{S}(8)$ $\cdots$
$\cdots$ $\mathbb{S}^2(-2)$	$\mathbb{S}^2(-1)$	$\mathbb{S}^2$	$\mathbb{S}^2(1)$	$\mathbb{S}^2(2)$	$\mathbb{S}^2(3)$	$\mathbb{S}^2(4)$	$\mathbb{S}^2(5)$	$\mathbb{S}^2(6)$	$\mathbb{S}^2(7)$	$\mathbb{S}^2(8)$ $\cdots$



$$\cdots\ \mathbb{C}(-2)\ \ \mathbb{C}(-1)\ \ \mathbb{C}\ \ \mathbb{C}(1)\ \ \mathbb{C}(2)\ \ \mathbb{C}(3)\ \ \mathbb{C}(4)\ \ \mathbb{C}(5)\ \ \mathbb{C}(6)\ \ \mathbb{C}(7)\ \ \mathbb{C}(8)\ \cdots$$

$$\cdots\ \mathcal{S}(-2)\ \ \mathcal{S}(-1)\ \ \mathcal{S}\ \ \mathcal{S}(1)\ \ \mathcal{S}(2)\ \ \mathcal{S}(3)\ \ \mathcal{S}(4)\ \ \mathcal{S}(5)\ \ \mathcal{S}(6)\ \ \mathcal{S}(7)\ \ \mathcal{S}(8)\ \cdots$$

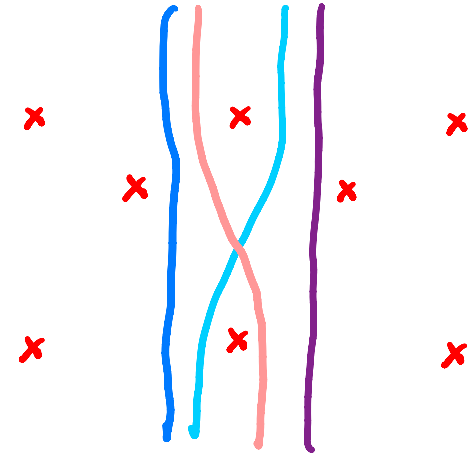
$$\cdots\ \mathcal{S}^2(-2)\ \ \mathcal{S}^2(-1)\ \ \mathcal{S}^2\ \ \mathcal{S}^2(1)\ \ \mathcal{S}^2(2)\ \ \mathcal{S}^2(3)\ \ \mathcal{S}^2(4)\ \ \mathcal{S}^2(5)\ \ \mathcal{S}^2(6)\ \ \mathcal{S}^2(7)\ \ \mathcal{S}^2(8)\ \cdots$$



$$\cdots\ \mathbb{C}(-2)\ \ \mathbb{C}(-1)\ \ \mathbb{C}\ \ \mathbb{C}(1)\ \ \mathbb{C}(2)\ \ \mathbb{C}(3)\ \ \mathbb{C}(4)\ \ \mathbb{C}(5)\ \ \mathbb{C}(6)\ \ \mathbb{C}(7)\ \ \mathbb{C}(8)\ \cdots$$

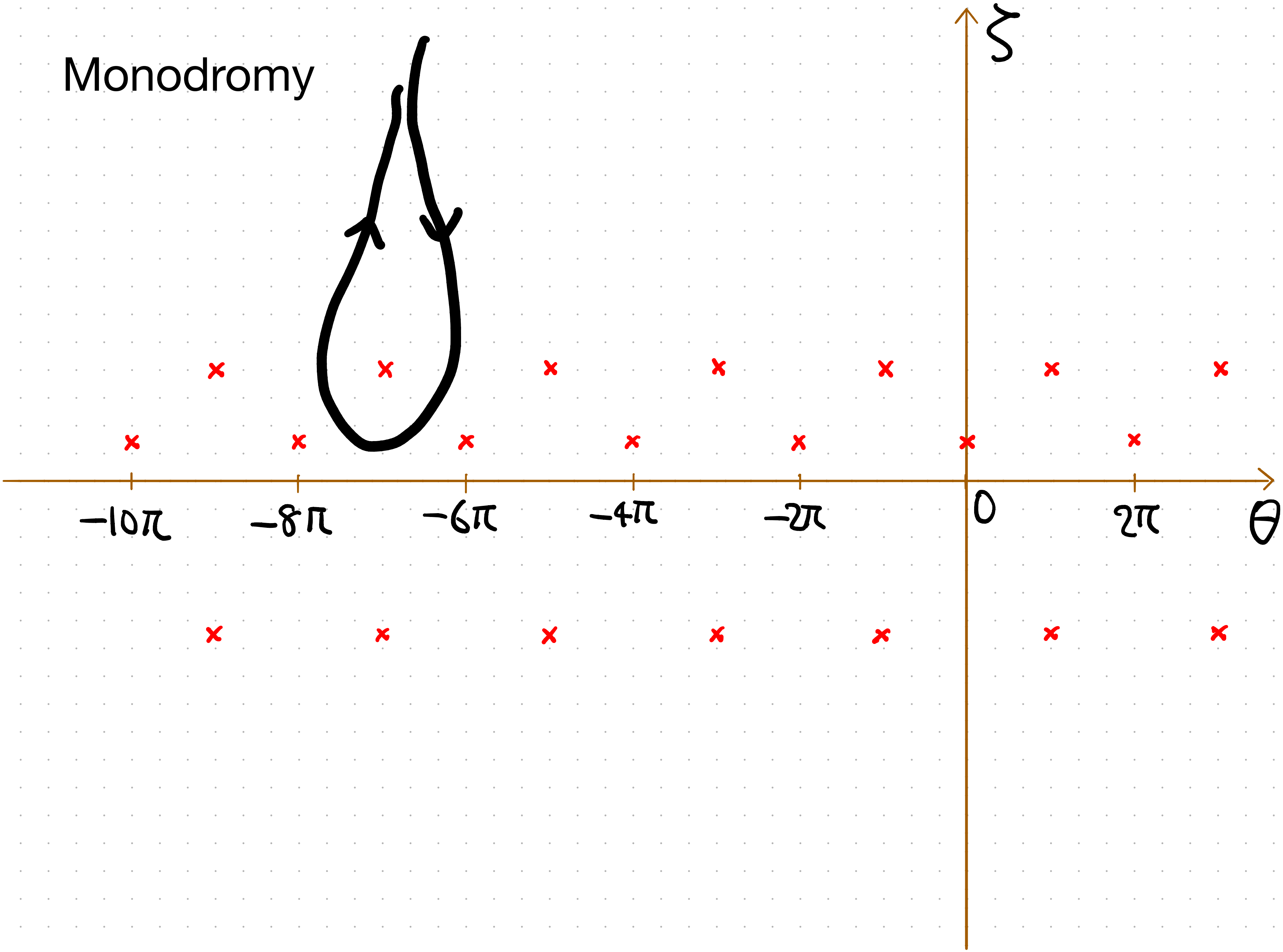
$$\cdots\ S(-2)\ \ S(-1)\ \ S\ \ S(1)\ \ S(2)\ \ S(3)\ \ S(4)\ \ S(5)\ \ S(6)\ \ S(7)\ \ S(8)\ \cdots$$

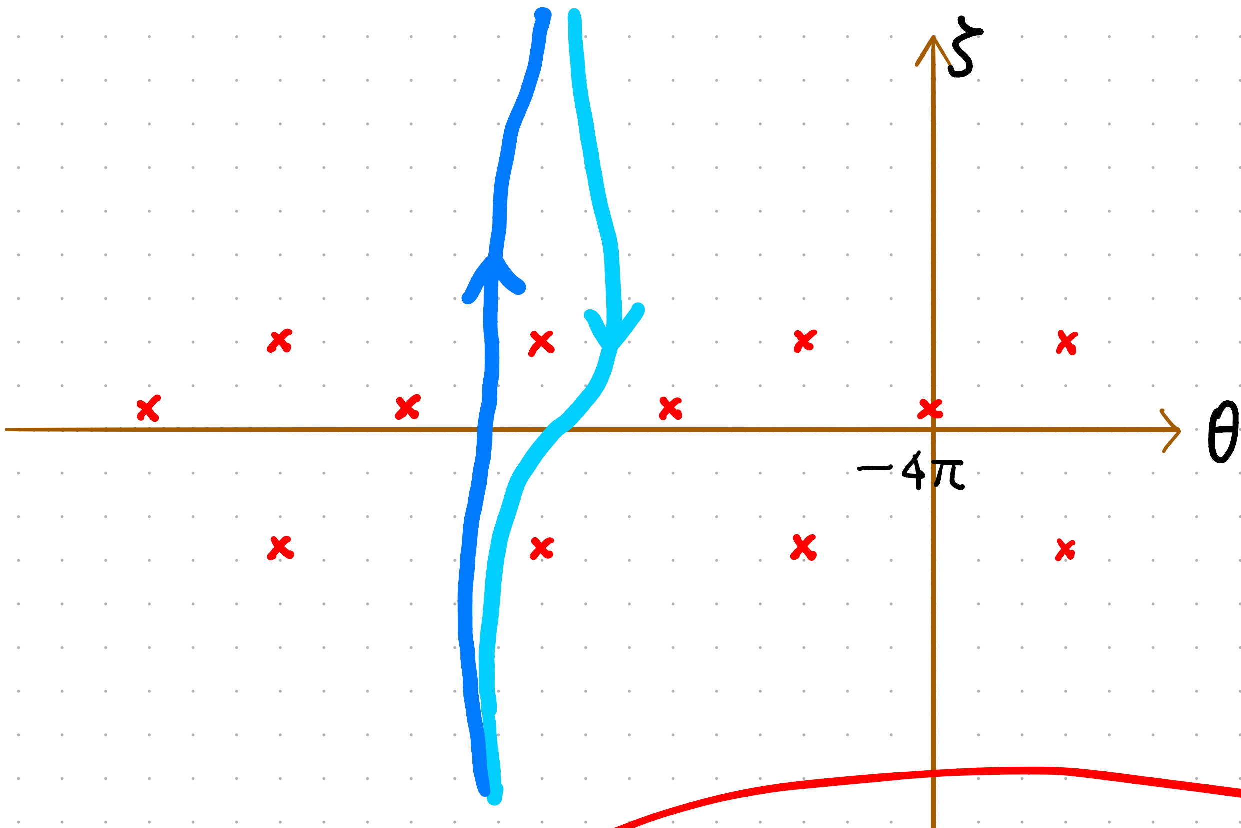
$$\cdots\ S^2(-2)\ \ S^2(-1)\ \ S^2\ \ S^2(1)\ \ S^2(2)\ \ S^2(3)\ \ S^2(4)\ \ S^2(5)\ \ S^2(6)\ \ S^2(7)\ \ S^2(8)\ \cdots$$

For each window $\mathbf{W}$ such as 

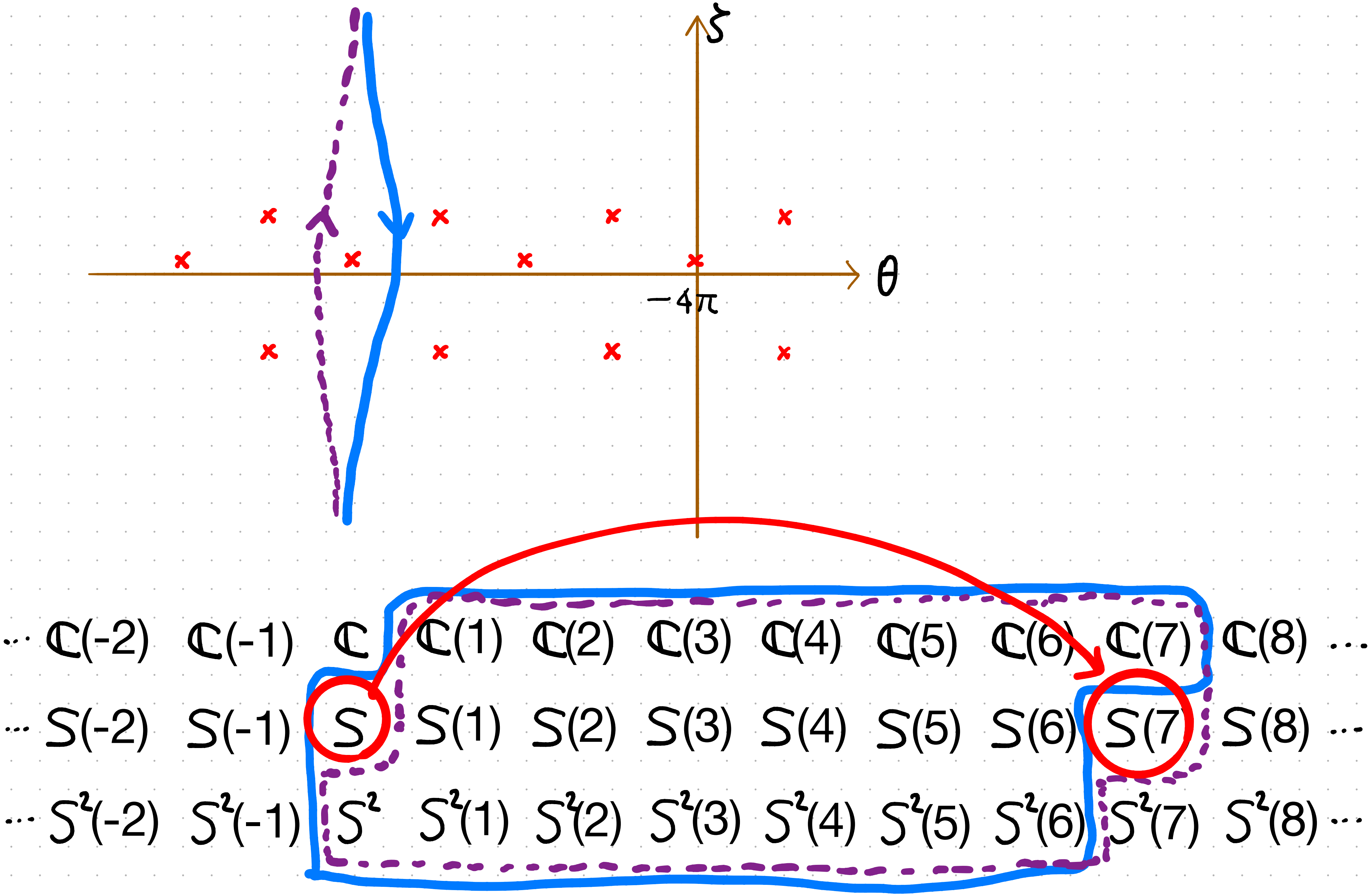
$$
\begin{tikzcd}
 & & MF_{U(2)}(W) \arrow[dddrr] & & \\
 & GR_{\leqslant 0} \arrow[ur, hook] \arrow[ddl] & & & \\
 & & GR_W \arrow[ul, hook] \arrow[dll, "\cong"'] \arrow[drr, "\cong"] & & \\
D^b(Y_A) & & & & D^b(X_A)
\end{tikzcd}
$$

Monodromy

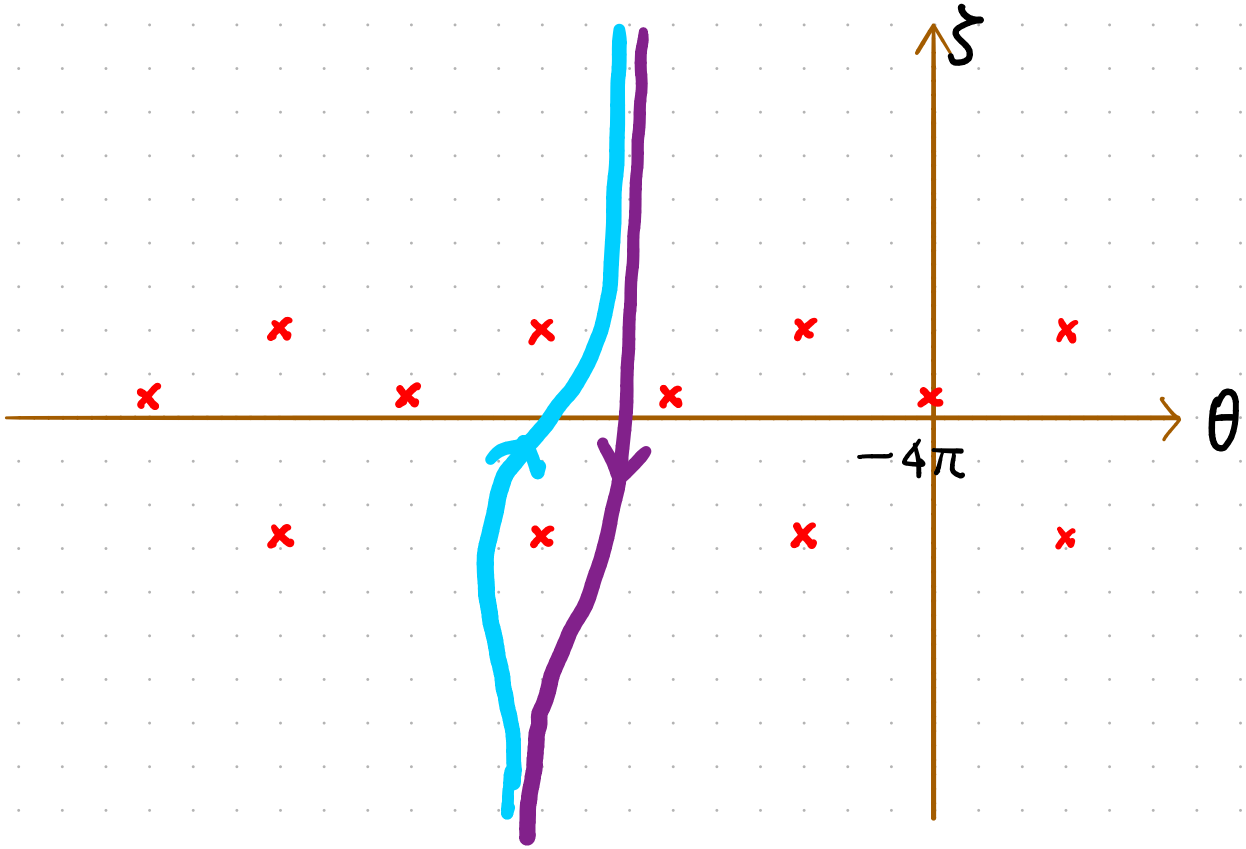




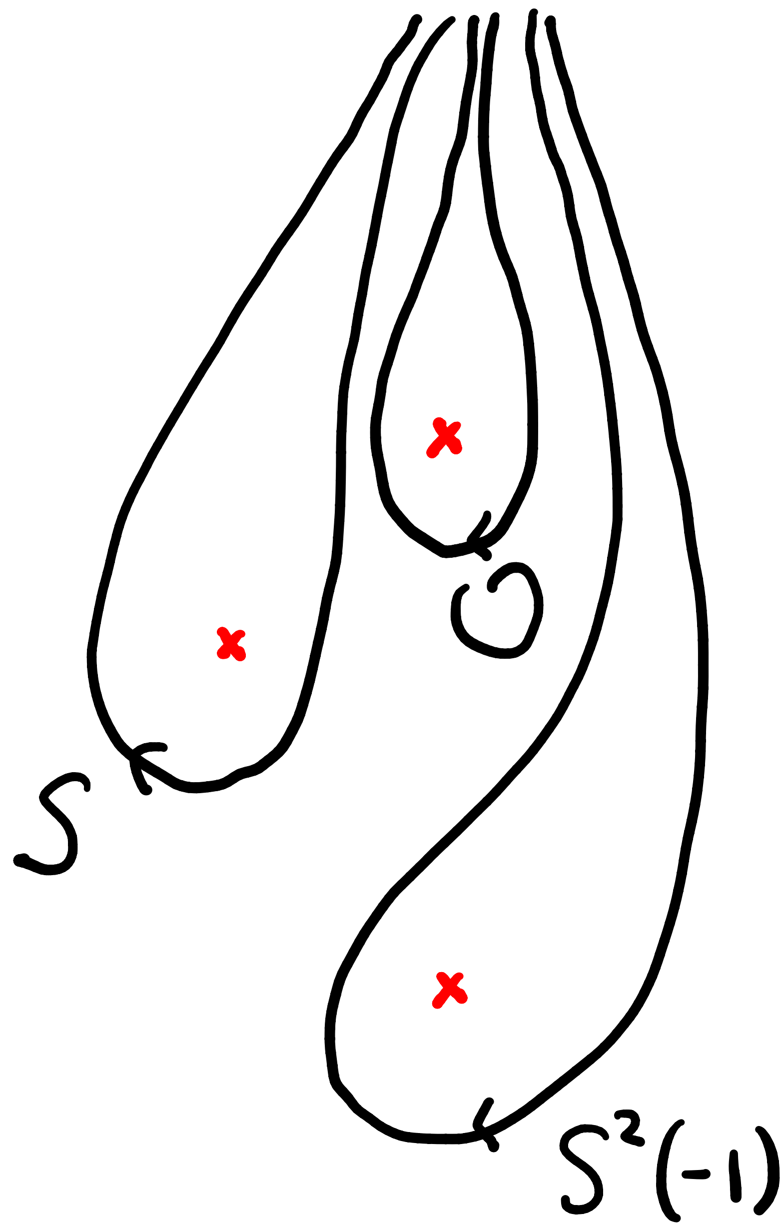
$\cdots$	$\mathbb{C}(-2)$	$\mathbb{C}(-1)$	$\mathbb{C}$	$\mathbb{C}(1)$	$\mathbb{C}(2)$	$\mathbb{C}(3)$	$\mathbb{C}(4)$	$\mathbb{C}(5)$	$\mathbb{C}(6)$	$\mathbb{C}(7)$	$\mathbb{C}(8)$	$\cdots$
$\cdots$	$S(-2)$	$S(-1)$	S	$S(1)$	$S(2)$	$S(3)$	$S(4)$	$S(5)$	$S(6)$	$S(7)$	$S(8)$	$\cdots$
$\cdots$	$S^2(-2)$	$S^2(-1)$	S^2	$S^2(1)$	$S^2(2)$	$S^2(3)$	$S^2(4)$	$S^2(5)$	$S^2(6)$	$S^2(7)$	$S^2(8)$	$\cdots$



ζ
θ
−4π
··· ℂ(-2) ℂ(-1) ℂ ℂ(1) ℂ(2) ℂ(3) ℂ(4) ℂ(5) ℂ(6) ℂ(7) ℂ(8) ···
··· S(-2) S(-1) S S(1) S(2) S(3) S(4) S(5) S(6) S(7) S(8) ···
··· S²(-2) S²(-1) S² S²(1) S²(2) S²(3) S²(4) S²(5) S²(6) S²(7) S²(8) ···



$\cdots$	$\mathbb{C}(-2)$	$\mathbb{C}(-1)$	$\mathbb{C}$	$\mathbb{C}(1)$	$\mathbb{C}(2)$	$\mathbb{C}(3)$	$\mathbb{C}(4)$	$\mathbb{C}(5)$	$\mathbb{C}(6)$	$\mathbb{C}(7)$	$\mathbb{C}(8)$	$\cdots$
$\cdots$	$S(-2)$	$S(-1)$	S	$S(1)$	$S(2)$	$S(3)$	$S(4)$	$S(5)$	$S(6)$	$S(7)$	$S(8)$	$\cdots$
$\cdots$	$S^2(-2)$	$S^2(-1)$	S^2	$S^2(1)$	$S^2(2)$	$S^2(3)$	$S^2(4)$	$S^2(5)$	$S^2(6)$	$S^2(7)$	$S^2(8)$	$\cdots$



Seidel-Thomas twists

These works are strongly motivated by mathematics.

HHP is motivated by Orlov 2005 $D^b(X_f) \cong MF_{\mathbb{Z}_5}(f)$.

EHKR is motivated by Addington-Donovan-Segal 2014

$$D^b(X_A) \cong D^b(Y_A).$$

There are also many other relevant works.

Halpern-Leistner, Ballard-Favero-Katzarkov 2012

Categorical VGIT.

Borisov-Caldararu, Kuznetsov 2006

first proof of $D^b(X_A) \cong D^b(Y_A)$.

Spenko-Van Den Bergh 2015

Non-commutative crepant resolution
of quotient singularities

Hosono-Takagi 2011

Donovan-Segal 2012

Halpern-Leistner-Sam 2016

Rennemo-Segal 2016

⋮

Kuznetsov 2005 Homological projective duality