

Algebras and Traces
at the boundary of 4d $N=4$ SYM

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Boundary operators are important: edge effects, feature in BCFT etc.

BPS operators at the SUSY boundaries: interesting math

3D $N=4$ theories have well-studied protected sectors

[MD-Fan-Pufu-Yacoby, Chester-Lee-Pufu-Yacoby, Beem-Peelaers-Rastelli, ...]

4D $N=4$ theories have well-studied SUSY boundary conditions [Gaiotto-Witten]

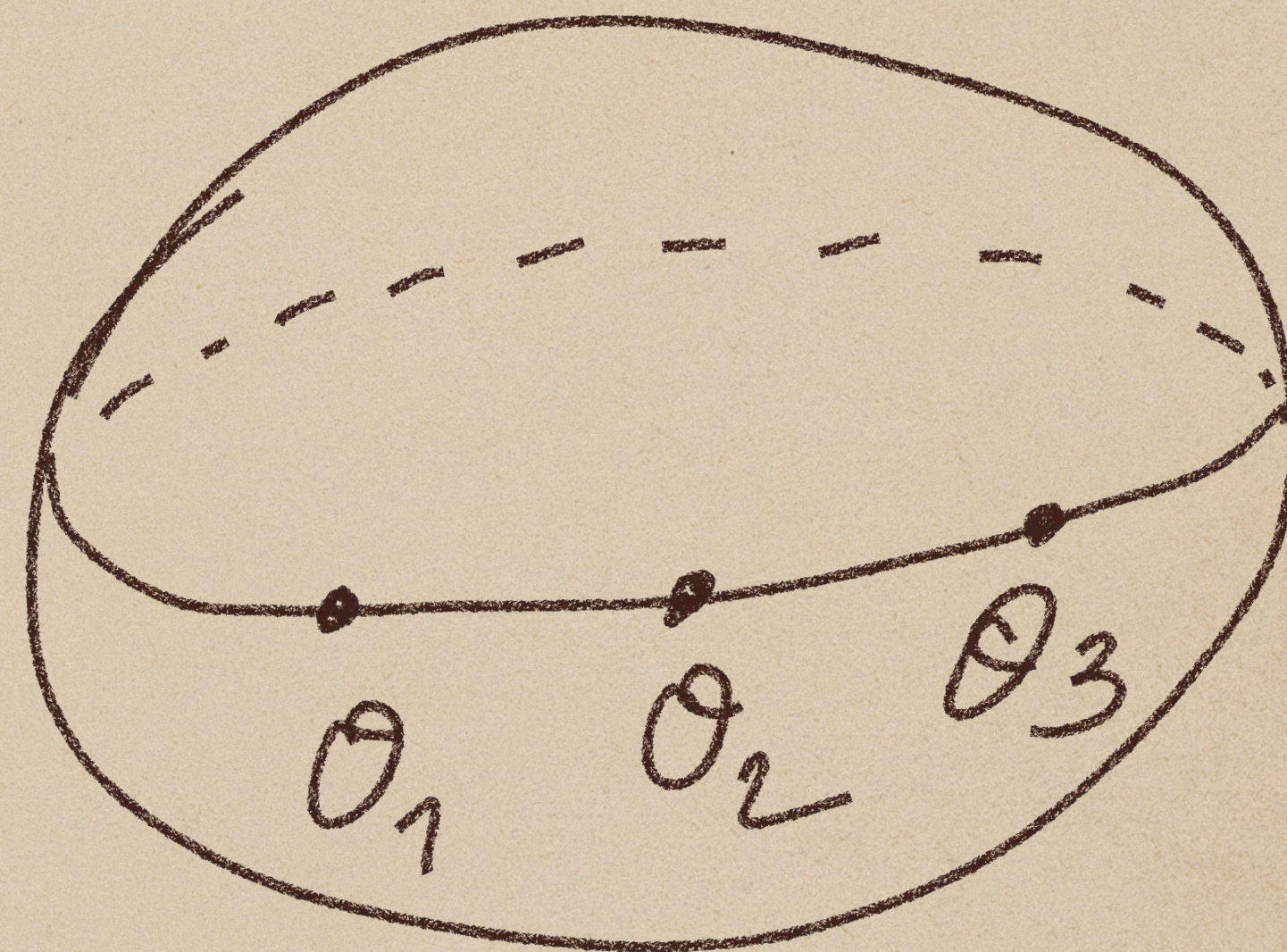
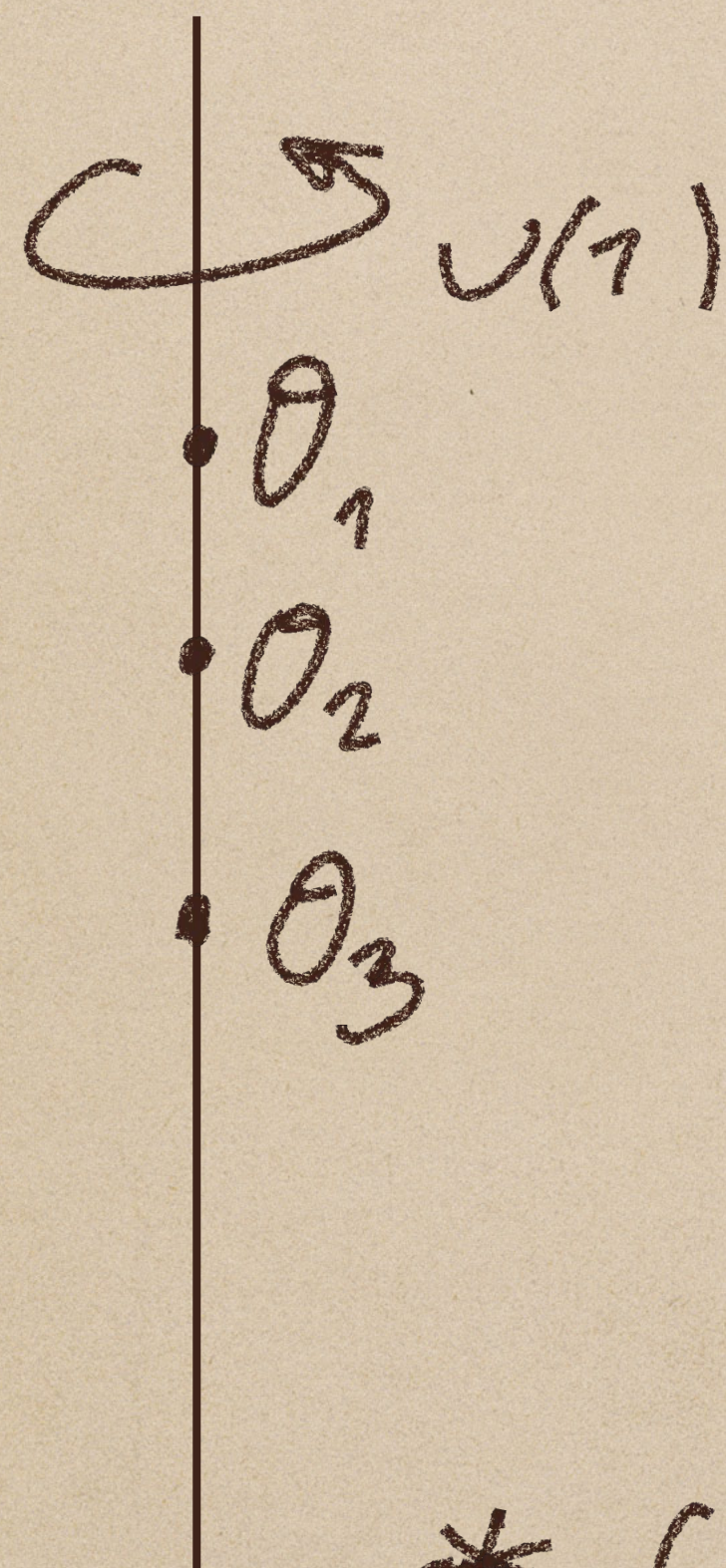
Combine the two subjects: protected sectors at half-BPS boundaries

[some recent progress by Wang and Komatsu]

A reminder

3d $\mathcal{N}=4$

S^3



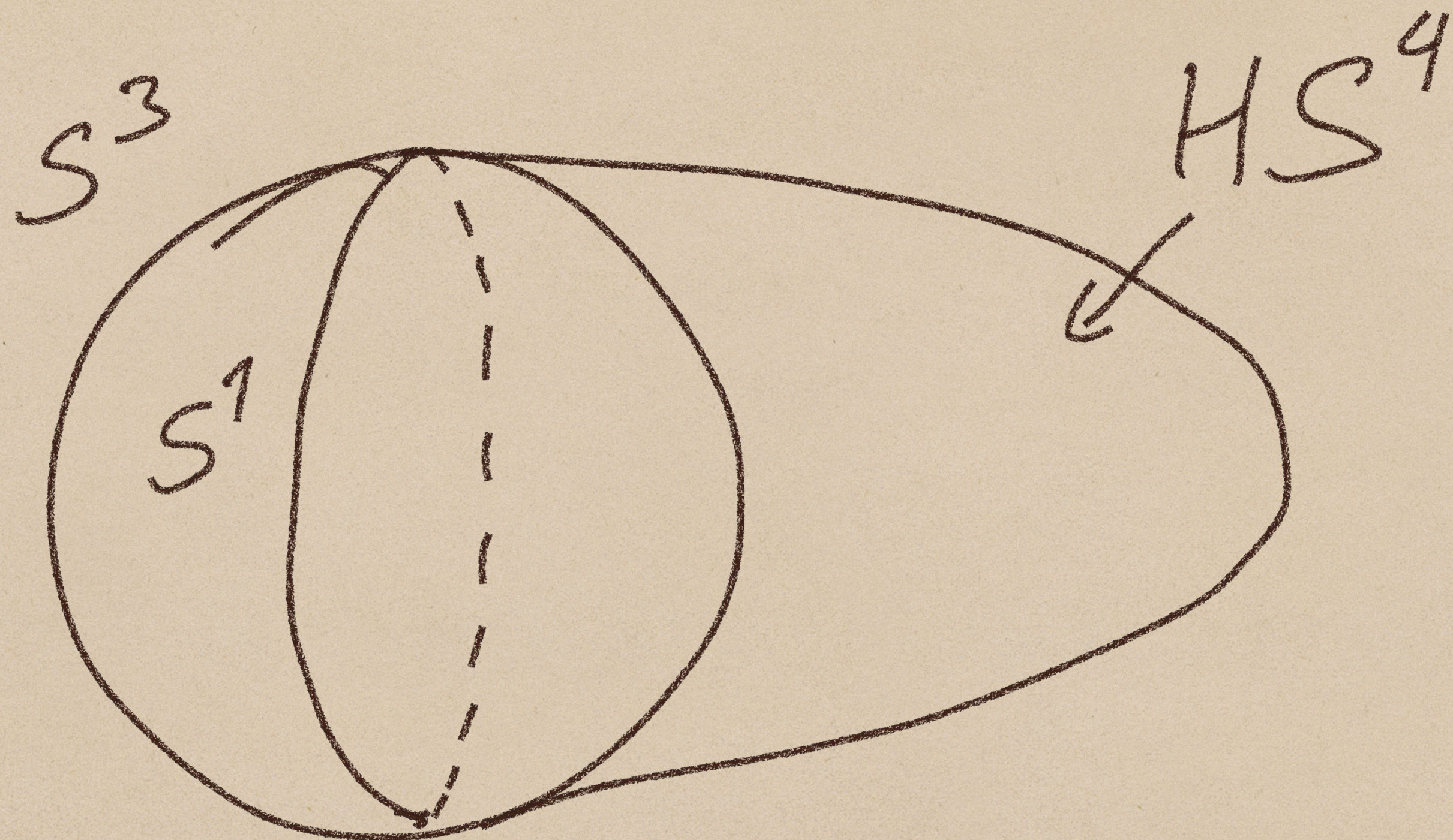
θ_i form assoc. algebra

Two of them:

* \mathcal{A}_H quantizes Higgs branch

* \mathcal{A}_C quantizes Coulomb branch

3D boundary of 4D $N=4$



$$S^1 = \partial(HS^2), \quad HS^2$$
$$\cap \quad \cap$$
$$S^3 = \partial(HS^4), \quad HS^4$$

* Pick supercharges Q_H & Q_C on HS^4

$Q^2 =$ rotation + R-symmetry fixes HS^2

Q-cohomology \simeq QFT on HS^2
(localization)

* Cohomology of \mathcal{Q}_H :

→ 2d constrained YM in the bulk (Pestun, Giombi, Wang)

→ 1d top. quant. mech. (TQM) @ boundary

* Cohomology of \mathcal{Q}_c :

→ 2d constrained YM in the bulk

→ 1d TQM @ boundary

???

$Q_H \rightarrow$ "electric construction"
2d c YM is related to electric variables

$Q_C \rightarrow$ "magnetic construction"
2d c YM is related to magnetic
(S-dual) variables.

Boundary for H: $A_H + \text{trace } T_H$

Boundary for C: $A_C + \text{trace } T_C$

Some facts:

- A_H is $\frac{1}{2} \mathbb{Z}$ -graded, degree $d \equiv R_H \equiv \text{conf. dim.}$
- T_H is a twisted trace: $T_H(xy) = T_H(e^{-1} \cdot yx)$
boundary mass $\xrightarrow{\quad}$ $\uparrow \uparrow \mathbb{Z}d \text{ mod } 2\mathbb{Z}$
- (A_H, T_H) encodes boundary correlators for electric 2d cYM

• A_c is $\frac{1}{2}\mathbb{Z}$ -graded, degree $d = \mathbb{R}_c = \text{cont. dim.}$

• T_c is twisted by $(-1)^{2\mathbb{R}_c}$ and boundary F.I. terms.

• (A_c, T_c) encodes boundary correlators of magnetic 2d c YM.

* In 3d, (A, T) encodes equivariant short
star-product on a hyper-Kähler cone
(Etingof-Stryker)

* In the 4d/3d system, (A, T) still encodes
quantization, BUT the underlying space
is Poisson \leftarrow moduli of vacua on $\mathbb{R}^3 \times \mathbb{R}_+$.

Example # 1

→ Dirichlet boundary conditions

⇒ Dirichlet b.c. in 2d cYM ⇒ pert. 2d BF on D^2
(w/ B @ ∂D^2)

⇒ Poisson sigma-model into $g_{\mathbb{C}}^*$

⇒ Kontsevich \star -product (Cattaneo-Felder)

\Rightarrow Quantization of $g_{\mathcal{C}}^*$ = $\mathcal{U}(g_{\mathcal{C}})$.

$A_H = \mathcal{U}(g_{\mathcal{C}})$ - algebra of boundary operators

Trace T_H is determined by its value on the center $Z[\mathcal{U}(g_{\mathcal{C}})]$
(follows from Ward id's)

Algebra of bulk operators: (in 2d YM)

$\mathcal{B}_H =$ gauge-inv. poly in $F_{\mu\nu} = F_{12}$
Can be thought of as $\mathbb{C}[g]^g \simeq \mathbb{C}[\#]^w$

Bulk-boundary map:

$$\rho_H: \mathcal{B}_H \longrightarrow \mathcal{A}_H; \quad \underline{\rho_H(\mathcal{B}_H) \subset \mathbb{Z}[\mathcal{A}_H]}$$

For $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, $\rho_H: \mathcal{B}_H \rightarrow \mathbb{Z}[A_H]$
is an isomorphism!

$$\mathbb{C}[\hbar]^W \cong \mathbb{Z}[\mathcal{U}(\mathfrak{g}_{\mathbb{C}})]$$

→ Harish-Chandra isomorphism

→ encodes physics of the map ρ_H

Trace can be expressed through traces on Verma modules V_g of $U(\mathfrak{g}_\mathbb{C})$:

$$T_H(\mathcal{O}) = \# \int [da] e^{-\frac{i\pi}{\tau} \text{Tr}(a^2)} \Delta(a) \text{Tr}_{V_{-ia-\rho}} \left(e^{-2\pi m \cdot \mathcal{B}} \sigma \right)$$

\uparrow boundary mass $\mathcal{B} \in \mathfrak{g}_\mathbb{C}$

\uparrow
 $\Delta(a) = \prod_{\alpha \in \Phi_+} \langle \alpha, a \rangle$

\swarrow
 GNO dual (or L)

• This is compatible with S-duality

• Generalizes conjecture of Gaiotto-Okazaki
expressing traces T_H & T_C as finite linear
combinations of traces on Verma modules in $3d$.

Here: $4d/3d$; continuous linear comb.

Example #2

→ Neumann. b.c. enriched by a 3d theory \mathcal{T} .

Equivalent description:

- 1) Take $[\text{Dirichlet}] \otimes \mathcal{T}$
- 2) Gauge $\text{Diag}(G \times G)$ via a 3D vector multiplet

3D gauging corresponds to quantum Hamiltonian reduction of A_H .

$$A_H = \left(\frac{A_H(\mathcal{T}) \otimes \mathcal{U}(\mathfrak{g}_\mathbb{C})}{(\mu)} \right)^{\mathfrak{g}}$$
$$= [A_H(\mathcal{T})]^{\mathfrak{g}}$$

A_c is obtained from $A_c(\mathcal{T})$
as a central extension:

$$0 \rightarrow \mathbb{C}[\hbar]^{\mathcal{W}} \xrightarrow{\text{bulk-boundary map}} A_c \longrightarrow A_c(\mathcal{T}) \longrightarrow 0$$

" A_c is obtained from $A_c(\mathcal{T})$ by promoting
masses to dynamical fields"

Writing trace is easy.

Skip to save some time.

Example # 3

→ Nahm pole b.c.

$X \sim \frac{\vec{t}}{y}$ ← $SU(2)$ triple
corresponding to
 $\rho: SU(2) \rightarrow \mathfrak{g}$

↙ Triplet of $SU(2)_H$

New challenges:

- $SU(2)_H$ R-symmetry mixes with gauge symmetry at the boundary
 \Rightarrow boundary R-charges are shifted.
- Singularity restricts boundary values of fields.
- Identification of boundary operators is interesting.

We find:

→ Space of boundary operators

moduli space of
Nahm's eqn's
on $\mathbb{R}^3 \times \mathbb{R}_+$

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Regular functions on Slodowy slice S_{t_+}

Fact: S_{t_+} has a natural Poisson structure; [Gan-Ginzburg]
Quantization \rightarrow finite \mathcal{W} algebra [I. Losev]

Conjecture

$A_H \{ \text{Nahm pole } \rho \} \simeq \text{finite } W\text{-algebra}$
 $W(\mathfrak{g}_{\mathbb{C}}, t_+)$

→ Reminder: ρ determines grading on $\mathfrak{g}_{\mathbb{C}}$,
 $\mathfrak{n} \subset \mathfrak{g}_{\mathbb{C}}$ - nilpotent subalgebra of $\text{deg} < 0$.

$W(\mathfrak{g}_{\mathbb{C}}, t_+)$ is roughly a quantum Hamiltonian reduction
of $U(\mathfrak{g}_{\mathbb{C}})$ over \mathfrak{n} .

The most convincing check:

S-dual: Neumann b.c. + $T_g[G^v]$

e.g. for $SU(N)$: $\bigcirc_{V_{k-1}} - \bigcirc_{V_{k-2}} - \dots - \bigcirc_{V_1} - \square_N$

This theory is known to have
(central quotient) of $\mathcal{W}(g_c, t_+)$ for its A_c .

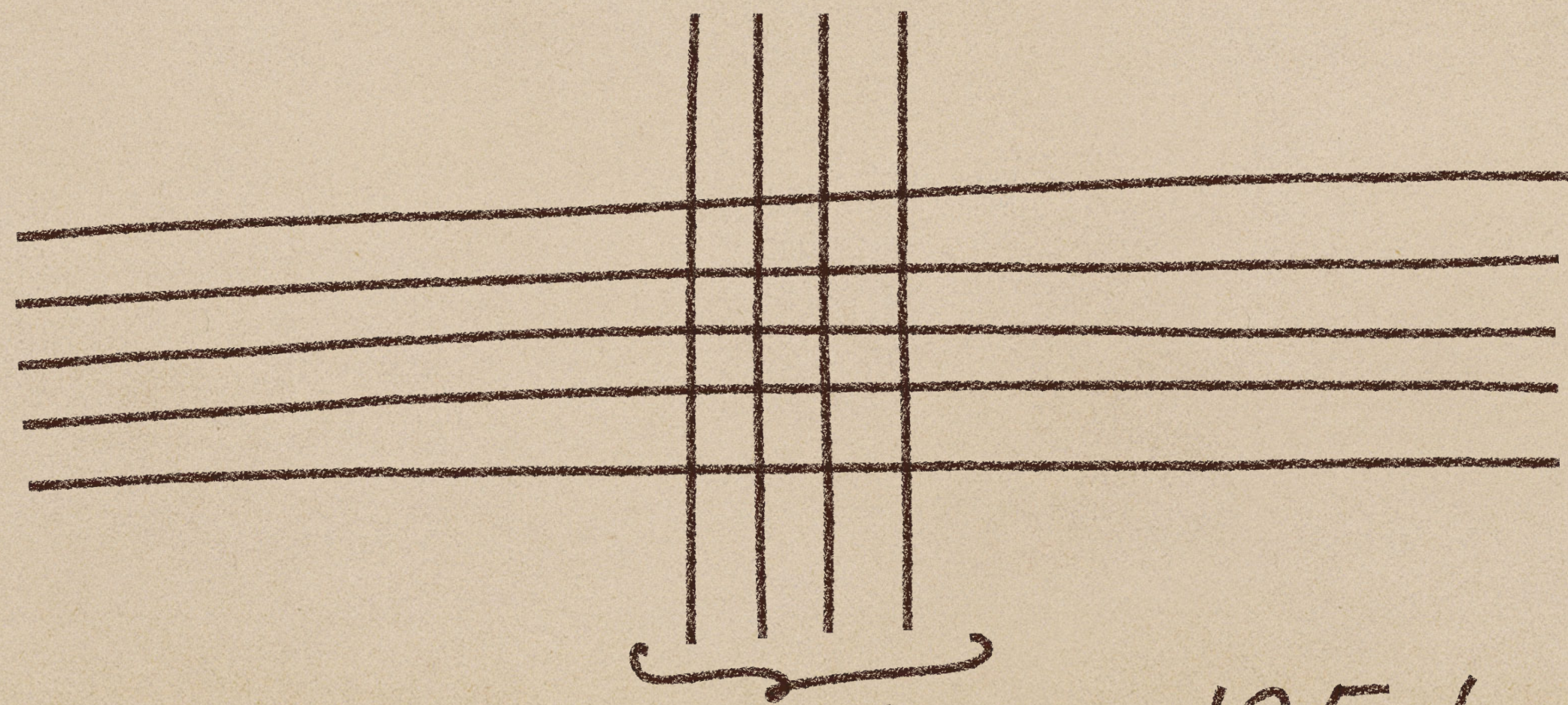
Can also write trace T_H as
a continuous linear combination
of Verma traces.

• $A_C[\text{Nahm pole}] \simeq \mathbb{C}$

Interfaces

(Example # 4)

N
D3 branes



n D5 or NS5 branes

$A_H(n \text{ D5's})$

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$A_C(n \text{ NS5's})$

$A_H(n \text{ D5's}) \equiv A_{N,n}$ admits description as
 quantum Hamiltonian reduction of:

$$U(\mathfrak{gl}_n) \otimes W^{N,n} \otimes U(\mathfrak{gl}_N)$$

Left D3 branes
 generators $(B_-)_{\alpha}^{\beta}$

\uparrow
 n fund. hypers
 at the interface
 Generators

\uparrow
 Right D3 branes
 generators $(B_+)_{\alpha}^{\beta}$

$X_{\alpha}^{\alpha}, Y_{\beta}^{\beta} \leftarrow$ Weyl algebra

There are two homomorphisms from the Yangian $Y(\mathfrak{gl}_n)$:

$$\bullet T[z] = \mathbb{1} - X \frac{1}{z - B_+} Y$$

$$\bullet T[z] = \mathbb{1} + X \frac{1}{z + B_-} Y$$

In the large- N limit, $A_{N,n}$ gets "closer" to $Y(\mathfrak{gl}_n)$

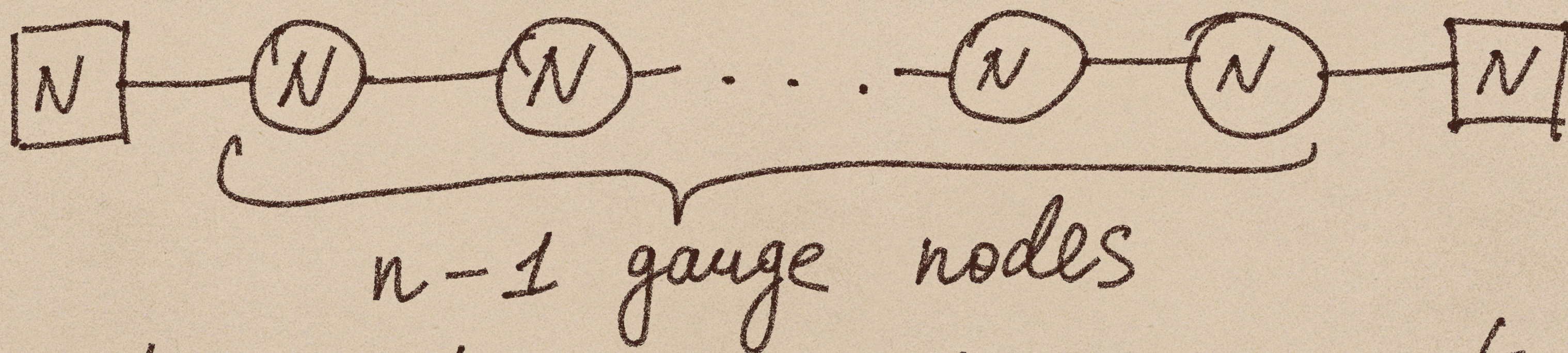
→ Trace T_H on $A_{N,n}$ induces traces on $Y(\mathfrak{gl}_n)$
via these homomorphisms.

Can we make them more explicit?

Yes, on the S -dual side.

Look at A_C (n NS5's).

Interface is engineered by a quiver:



→ Construct its A_C , promote masses to fields.

The algebra itself was constructed
e.g. in [Bullimore-Dimofte-Gaiotto], [Braverman-Finkelberg
-Nakajima]

To compute the trace, have to use [MD-Fan-Pufu-Yacoby]

↳ this yields representation of [Gerasimov-Kharchev-
-Lebedev-Oblezin]
of the Yangian in terms of difference operators.

The final answer for $\langle T[z_1]_{a_1}^{b_1} T[z_2]_{a_2}^{b_2} \dots T[z_L]_{a_L}^{b_L} \rangle$

is conjecturally related to

a length- L sl_n XXX spin chain

→ Twisted traces form a commutative algebra.

• Traces over evaluation modules (of fin.-dim. irreps of sl_n) can be taken as a generating set.

• Alternatively, traces over evaluation modules for Vermas.

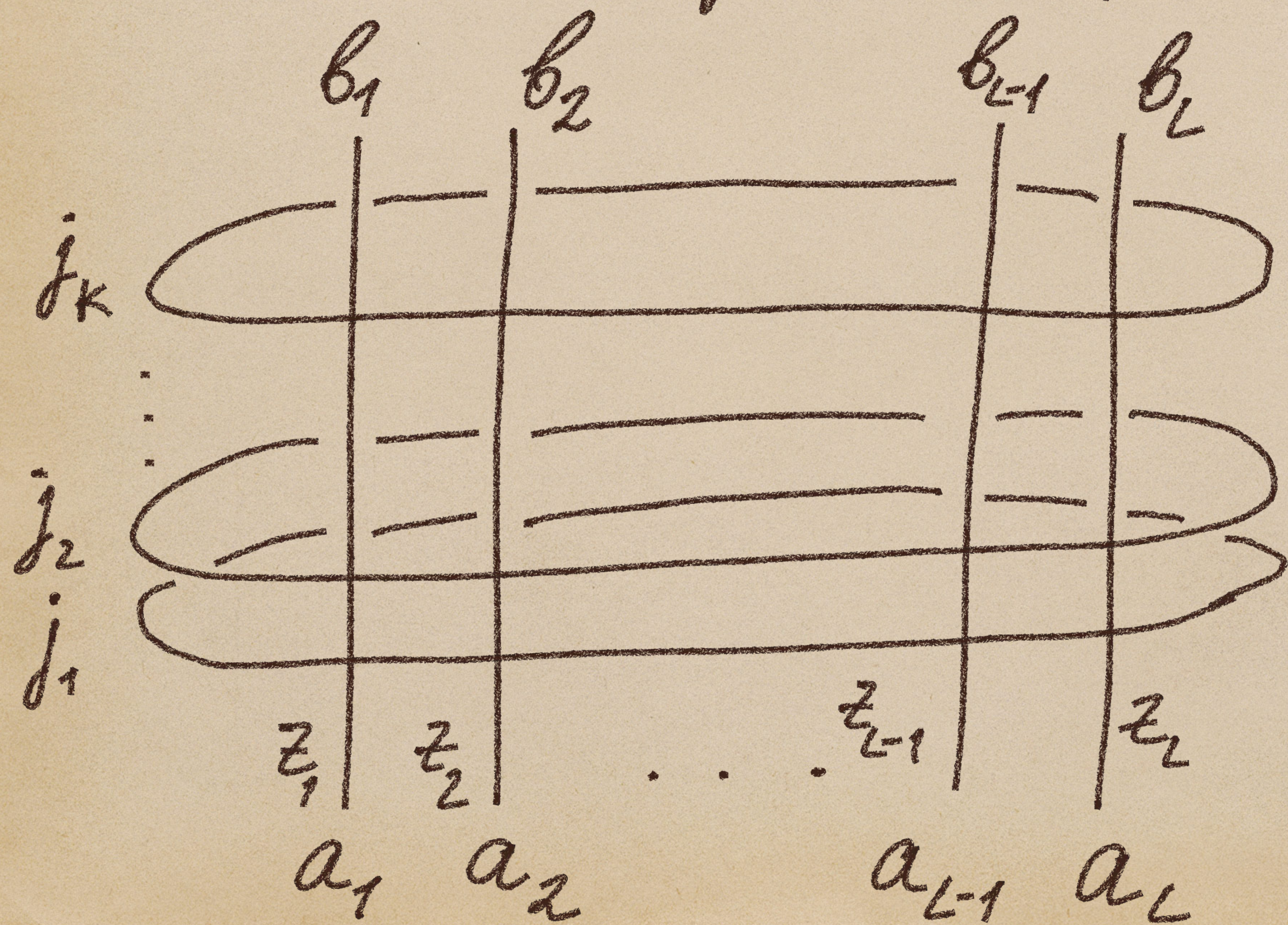
L-operator: $L(z) = z \cdot \mathbb{1} + \sum E_{ij} \otimes J_{ij}$ elementary sl_n matrices
generators of sl_n

Satisfies: $R(x-y)(L(x) \otimes \mathbb{1})(\mathbb{1} \otimes L(y)) = (\mathbb{1} \otimes L(y))(L(x) \otimes \mathbb{1})R(x-y)$

$R(z) = z + P: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}^n$; $L(z)$ determines eval. module

Pictorially: $\mathbb{L}(z)_a^b = \begin{array}{c} b \\ | \\ \text{---} j \\ | \\ z \\ | \\ a \end{array}$, j labels sl_n rep.

Twisted trace of $T[z_1]_{a_1}^{b_1} \dots T[z_L]_{a_L}^{b_L}$ in $[j_1] \otimes \dots \otimes [j_k]$:



Matrix element of a product of k transfer matrices of a length- L periodic inhomogeneous sl_n spin chain.

Just need to determine representations...

So far: $\langle T[z_1]_{a_1}^{b_1} \dots T[z_L]_{a_L}^{b_L} \rangle$

\Downarrow
products of transfer matrices T
in a length- L sl_n spin chain

n = number of fivebranes

Consider $n=2$.

Using techniques of [MD-Fan-Pufu-Yacoby], can
directly compute $\langle T[z_1]_{a_1}^{b_1} \dots T[z_L]_{a_L}^{b_L} \rangle \dots$

It turns out that...

The answer involves N sl_2 Verma modules.

I.e., $[j_1], \dots, [j_N]$ — eval. modules corresp. to
 sl_2 Verma modules of h.w. j_m .

The correlator is:

$$\sum_{\substack{\sigma \in \frac{S_{2N}}{S_N \times S_N}}} \frac{z^{-N(2N+1)}}{\prod_{a=1}^N \prod_{k=N+1}^{2N} 2 \sinh \pi (\mu_{\sigma(a)} - \mu_{\sigma(k)})} \langle a_1 \dots a_L | \prod_{a=1}^N T^{\dagger}(\vec{z}) | b_1 \dots b_L \rangle$$

Transfer matrix \uparrow
 in eval. Verma, h.w. = $-\frac{1}{2} - \mu_{\sigma(a)}$

$$T^{\dagger}_{-\frac{1}{2}-\mu}(\vec{z}) = \frac{1}{2 \sinh \pi z} Q_+(\vec{z} - \mu \cdot \vec{1}) Q_-(\vec{z} + \mu \cdot \vec{1})$$

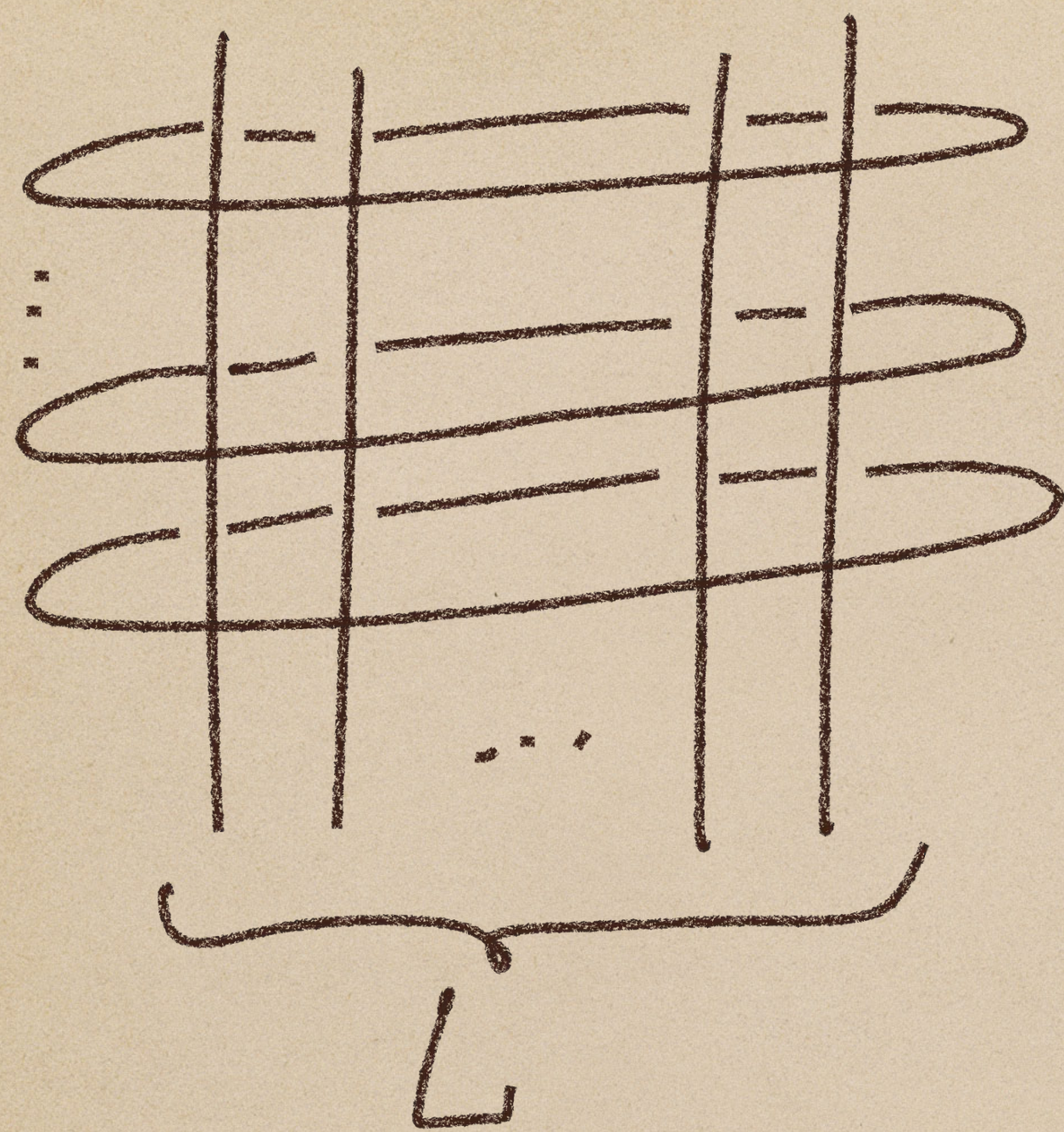
\uparrow
 \uparrow
 Baxter Q-operators

Remarkably explicit answer!

Q_{\pm} can be computed algorithmically
using the results of [Bazhanov-Lukowski]
(as traces over an auxiliary oscillator Fock space) [-Meneghelli-Staudacher]

Compute Q_{\pm} for each $L \Rightarrow$ answers for $\forall N$

Can do large- N etc...



$N \leftarrow$ the same as
of D3 branes!

To do: generalize to $n \geq 2$

For the nearest future: holography.

Thank You!