Algebras and Traces at the boundary of 4d N=4 SYM

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Boundary operators are important: edge effects, feature in BCFT etc.

BPS operators at the SUSY boundaries: interesting math

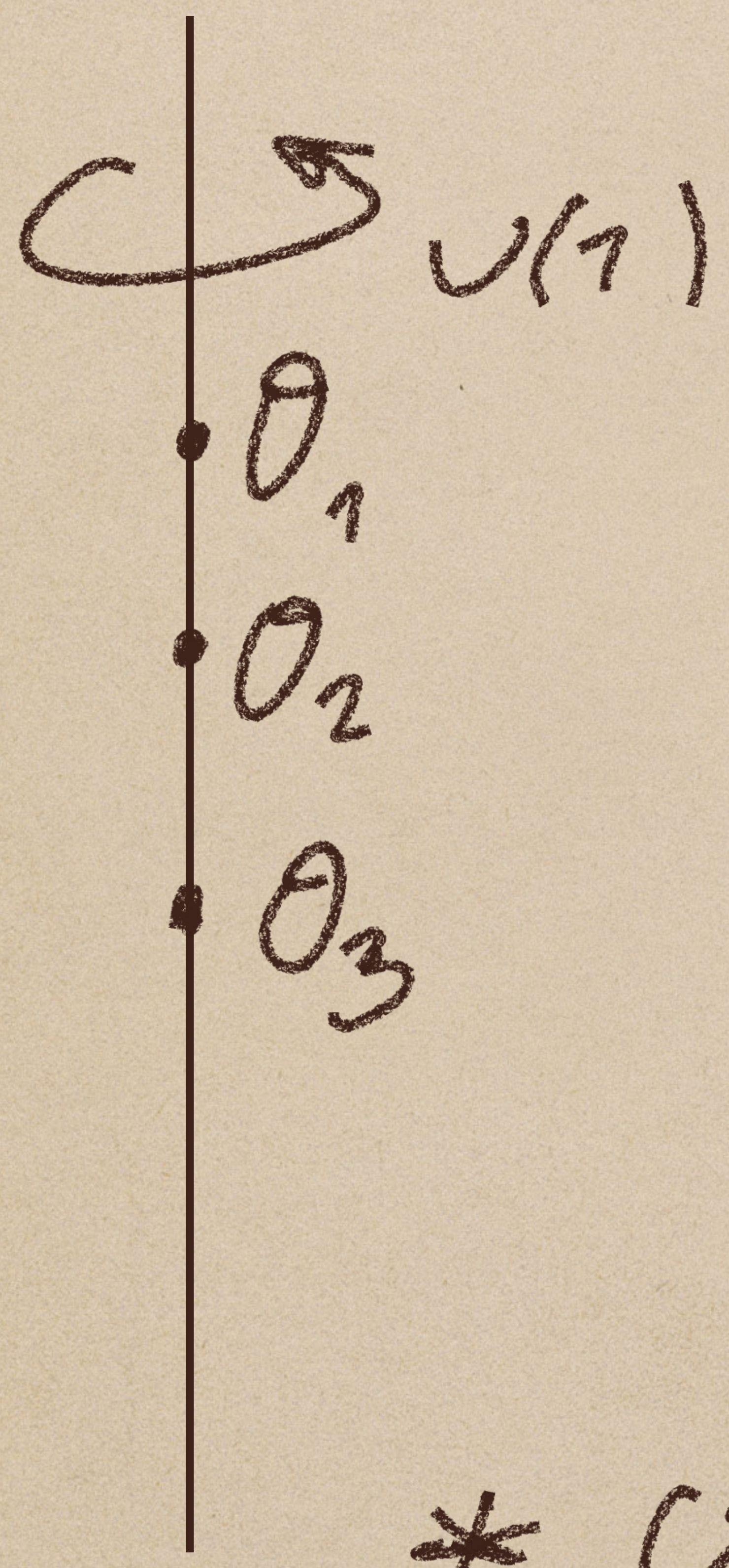
3D N=4 theories have well-studied protected sectors
[MD-Fan-Pufu-Yacoby,Chester-Lee-Pufu-Yacoby,Beem-Peelaers-Rastelli,...]

4D N=4 theories have well-studied SUSY boundary conditions [Gaiotto-Witten]

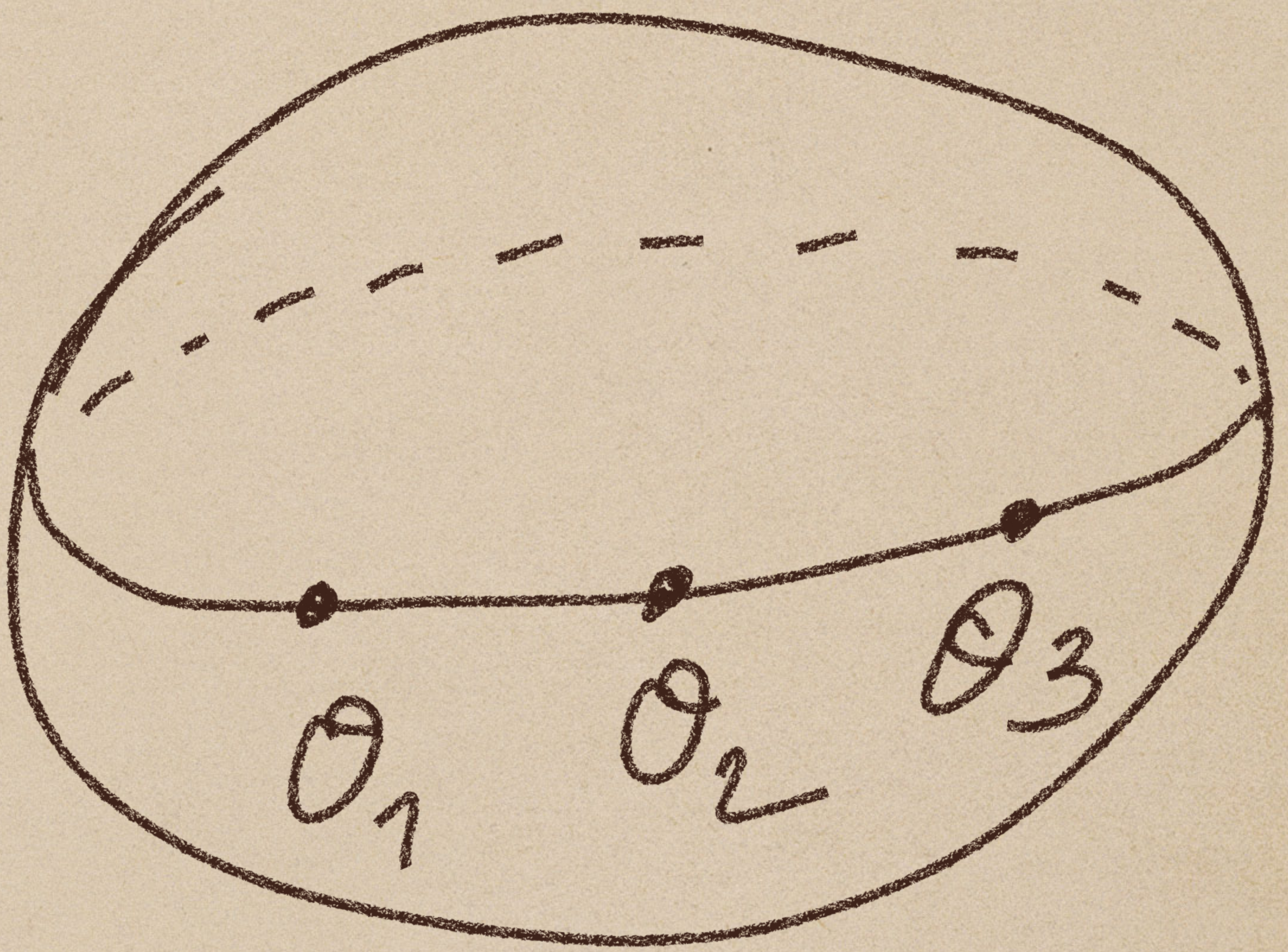
Combine the two subjects: protected sectors at half-BPS boundaries
[some recent progress by Wang and Komatsu]

A reminder

$3d\ \mathcal{N}=4$



S^3

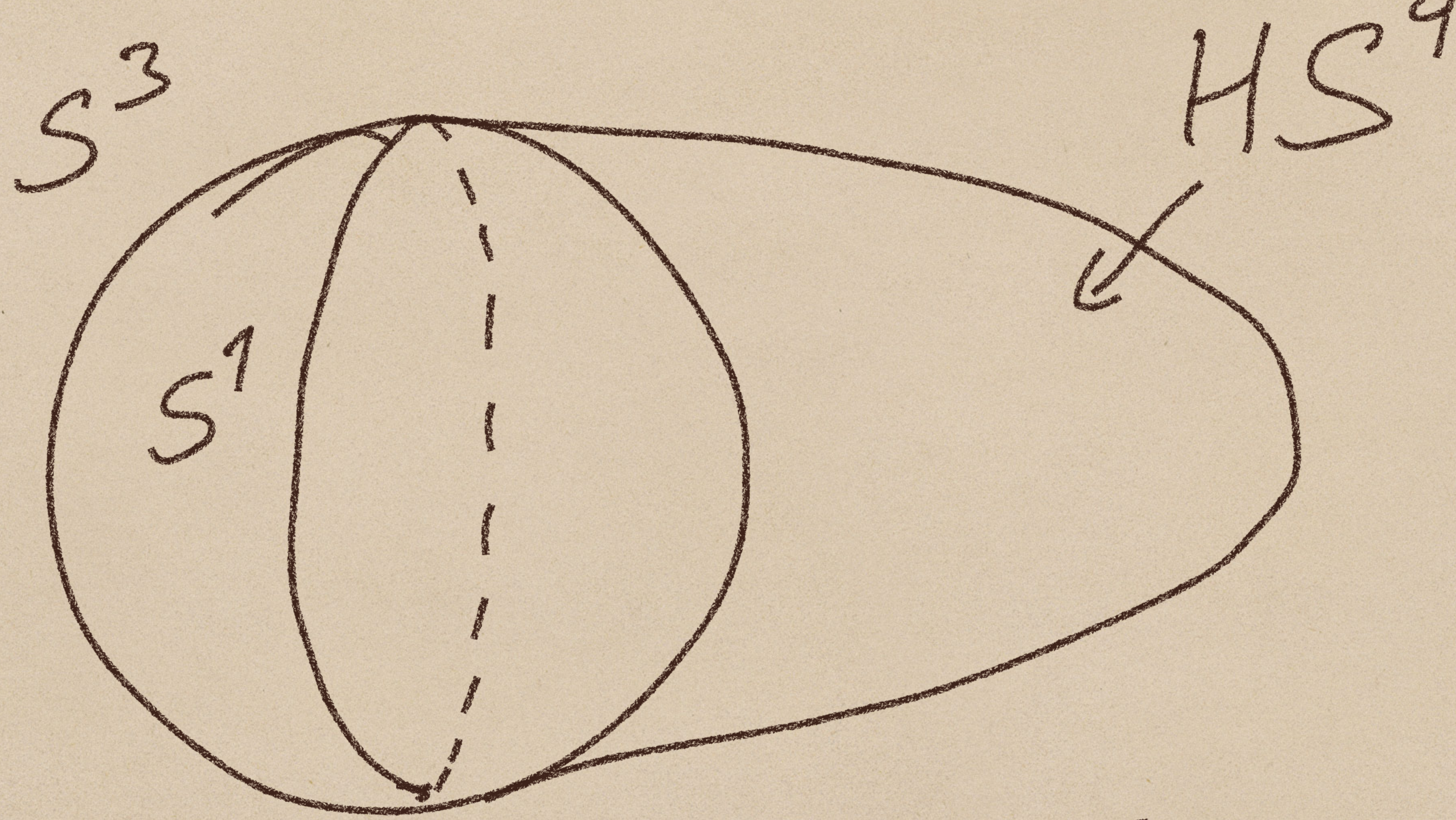


$\mathcal{O}_i$ form assoc. algebra

Two of them:

* $\mathcal{A}_H$ quantizes Higgs branch
* $\mathcal{A}_C$ quantizes Coulomb branch

3D boundary of 4D N=4



$$\begin{array}{ll} S^1 = \partial(HS^2), & HS^2 \\ \cap & \cap \\ S^3 = \partial(HS^4), & HS^4 \end{array}$$

* Pick supercharges Q_H & Q_C on HS^4

$Q^2 = \underbrace{\text{rotation} + \text{R-symmetry}}$ fixes HS^2

Q-cohomology $\simeq$ QFT on HS^2

(localization)

* Cohomology of Q_H:

$\longrightarrow$ 2d constrained YM in the bulk (Pestun, Giombi, Wang)

$\rightarrow$ 1d top. quant. mech. (TQM) @ boundary

* Cohomology of Q_C:

$\rightarrow$ 2d constrained YM in the bulk

$\rightarrow$ 1d TQM @ boundary

???

$Q_H \longrightarrow$ "electric construction"

2d cYM is related to electric variables

$Q_C \longrightarrow$ "magnetic construction"

2d cYM is related to magnetic (S-dual) variables.

Boundary for H: $\mathcal{A}_H$ + trace T_H

Boundary for C: $\mathcal{A}_C$ + trace T_C

Some facts:

- $\mathcal{A}_H$ is $\frac{1}{2}\mathbb{Z}$-graded, degree $d = R_H =$ conf. dim.

- T_H is a twisted trace: $T_H(xy) = T_H\Big(\underbrace{e^{-m}}_{\text{boundary mass}}\underbrace{(-1)^{2R_H}}_{2d \bmod 2\mathbb{Z}}\cdot y\, x\Big)$

- $(\mathcal{A}_H, T_H)$ encodes boundary correlators for electric 2d cYM

- $\mathcal{A}_c$ is $\frac{1}{2}\mathbb{Z}$-graded, degree $d = R_c$ = conf. dim.

- T_c is twisted by $(-1)^{2R_c}$ and boundary F.I. terms.

- $(\mathcal{A}_c, T_c)$ encodes boundary correlators of magnetic 2d c YM.

* In 3d, $(\mathcal{A}, T)$ encodes equivariant short star-product on a hyper-Kähler cone (Etingof-Stryker)

* In the 4d/3d system, $(\mathcal{A}, T)$ still encodes quantization, BUT the underlying space is Poisson $\leftarrow$ moduli of vacua on $\mathbb{R}^3 \times \mathbb{R}_+$.

Example #1

$\rightarrow$ Dirichlet boundary conditions

$\Rightarrow$ Dirichlet b.c. in 2d cYM $\Rightarrow$ pert. 2d BF on $\mathbb{D}^2$ (w/ B @ $\partial \mathbb{D}^2$)

$\Rightarrow$ Poisson sigma-model into $\mathfrak{g}^*_{\mathbb{C}}$

$\Rightarrow$ Kontsevich $\star$-product (Cattaneo-Felder)

$\Longrightarrow$ Quantization of $\mathfrak{g}_{\mathbb{C}}^{*} = U(\mathfrak{g}_{\mathbb{C}})$.

$\mathcal{A}_H = U(\mathfrak{g}_{\mathbb{C}})$ – algebra of boundary operators

Trace T_H is determined by its value on the center $Z[U(\mathfrak{g}_{\mathbb{C}})]$ (follows from Ward id's)

Algebra of bulk operators: (in 2d YM)

$\mathcal{B}_H$ = gauge-inv. poly in $F_{\mu\nu} : F_{12}$

Can be thought of as $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{t}]^{W}$

Bulk-boundary map:

$\rho_H : \mathcal{B}_H \longrightarrow \mathcal{A}_H$; $\underline{\rho_H(\mathcal{B}_H) \subset Z[\mathcal{A}_H]}$

For $U(\mathfrak{g}_{\mathbb{C}})$, $\rho_H : \mathcal{B}_H \longrightarrow Z[\mathcal{A}_H]$

is an isomorphism!

$$\mathbb{C}[\mathfrak{t}]^W \simeq Z[U(\mathfrak{g}_{\mathbb{C}})]$$

$\rightarrow$ Harish-Chandra isomorphism

$\rightarrow$ encodes physics of the map ρ_H

Trace can be expressed through traces on Verma modules $\overline{V}_\lambda$ of $U(\mathfrak{g}_{\mathbb{C}})$:

$$T_H(\mathcal{O}) = \# \int_{\mathfrak{t}^\vee} [da]\, e^{-\frac{i\pi}{\tau}\mathrm{Tr}(a^2)}\, \Delta(a)\, \mathrm{Tr}_{\overline{V}_{-ia-\rho}}\left(e^{-2\pi m \cdot B}\, \mathcal{O}\right)$$

GNO dual (or L)

$$\Delta(a) = \prod_{\alpha \in \Phi_+} \langle \alpha, a \rangle$$

boundary mass

$B \in \mathfrak{g}_{\mathbb{C}}$

- This is compatible with S-duality

- Generalizes conjecture of Gaiotto-Okazaki expressing traces T_H & T_C as finite linear combinations of traces on Verma modules in 3d.

Here: 4d/3d ; continuous linear comb.

Example #2

$\rightarrow$ Neumann. b.c. enriched by a 3d theory $\mathcal{T}$.

Equivalent description:

1) Take $[\text{Dirichlet}] \otimes \mathcal{T}$

2) Gauge $\text{Diag}(G \times G)$ via a 3D vector multiplet

3D gauging corresponds to quantum Hamiltonian reduction of $\mathcal{A}_H$.

$$\mathcal{A}_H = \left(\mathcal{A}_H(T) \otimes U(\mathfrak{g}_{\mathbb{C}}) \big/ (\mu) \right)^{\mathfrak{g}}$$

$$= \left(\mathcal{A}_H(T) \right)^{\mathfrak{g}}$$

$\mathcal{A}_C$ is obtained from $\mathcal{A}_C(\mathcal{T})$
as a central extension:

$$0 \to \mathbb{C}[\![t]\!]^W \xrightarrow[\downarrow]{\text{bulk-boundary map}} \mathcal{A}_C \longrightarrow \mathcal{A}_C(\mathcal{T}) \longrightarrow 0$$

"$\mathcal{A}_C$ is obtained from $\mathcal{A}_C(\mathcal{T})$ by promoting
masses to dynamical fields"

Writing trace is easy.

Skip to save some time.

Example # 3

$\longrightarrow$ Nahm pole b. c.

$$\vec{X} \sim \frac{\vec{t}}{y}$$

$\vec{t}$ ← $su(2)$ triple corresponding to

$$\rho : su(2) \longrightarrow \mathfrak{g}$$

$\vec{X}$ ← Triplet of $SU(2)_H$

New challenges:

- $SU(2)_H$ R-symmetry mixes with gauge symmetry at the boundary
  $\Rightarrow$ boundary R-charges are shifted.
- Singularity restricts boundary values of fields.
- Identification of boundary operators is interesting.

We find:

moduli space of Nahm's eqn's on $\mathbb{R}^3 \times \mathbb{R}_+$

$\rightarrow$ $\boxed{\text{Space of boundary operators}}$

$\cong$

$\boxed{\text{Regular functions on Slodowy slice } S_{t_+}}$

Fact: S_{t_+} has a natural Poisson structure, [Gan-Ginzburg, I. Losev]

quantization $\longrightarrow$ finite $\mathcal{W}$ algebra

Conjecture

$\mathcal{A}_H\{\text{Nahm pole } \rho\} \simeq$ finite W-algebra $\mathcal{W}(\mathfrak{g}_{\mathbb{C}}, t_+)$

→ Reminder: ρ determines grading on $\mathfrak{g}_{\mathbb{C}}$,
$\mathfrak{n} \subset \mathfrak{g}_{\mathbb{C}}$ – nilpotent subalgebra of deg < 0.

$\mathcal{W}(\mathfrak{g}_{\mathbb{C}}, t_+)$ is roughly a quantum Hamiltonian reduction of $U(\mathfrak{g}_{\mathbb{C}})$ over $\mathfrak{n}$.

The most convincing check:

S-dual: Neumann b.c. + $T_\rho[G^\vee]$

e.g. for $SU(N)$: $v_{k-1} - v_{k-2} - \cdots - v_1 - \boxed{N}$

This theory is known to have (central quotient) of $\mathcal{W}(\mathfrak{g}_{\mathbb{C}}, t_+)$ for its $\mathcal{A}_C$.

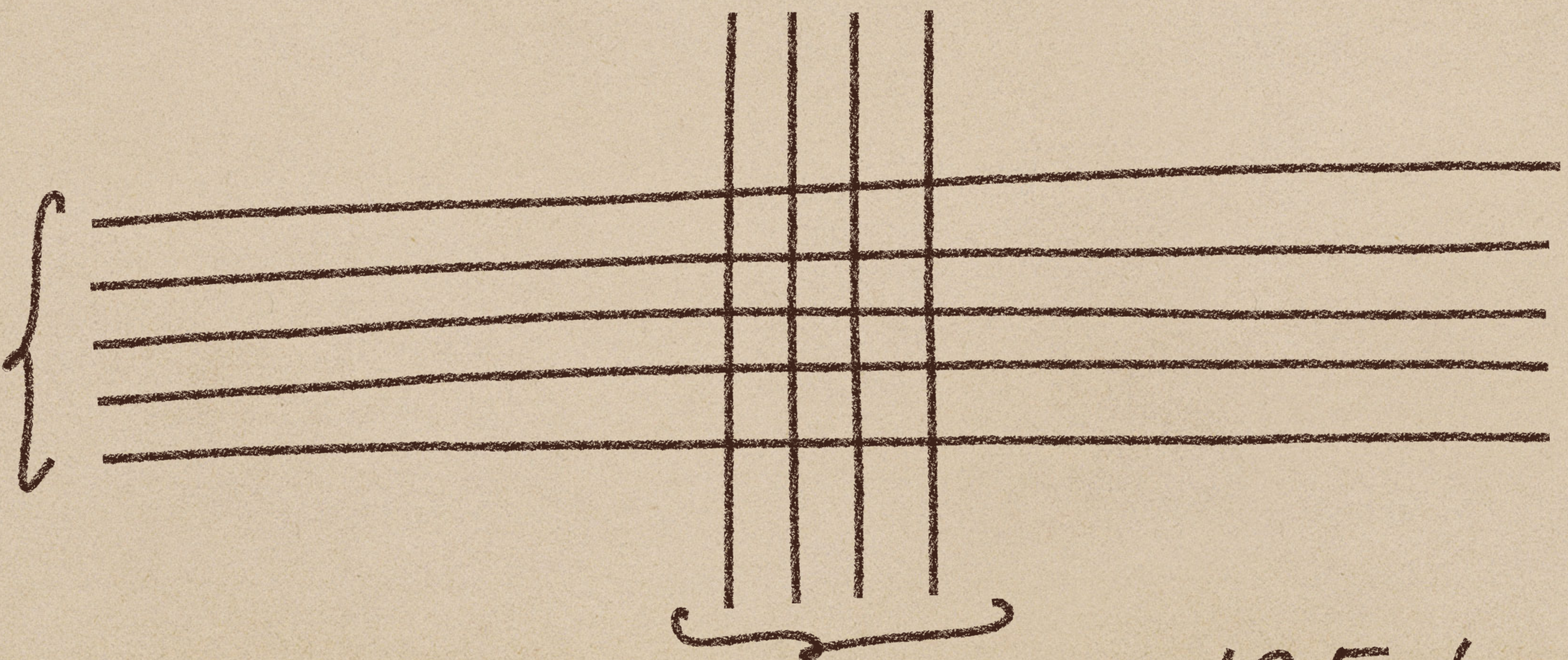
Can also write trace T_H as a continuous linear combination of Verma traces.

- $\mathcal{A}_{\mathbb{C}}[\text{Nahm pole}] \simeq \mathbb{C}$

Interfaces

(Example # 4)

N D3 branes



n D5 or NS5 branes

$$\mathcal{A}_H(n \text{ D5's}) \cong \mathcal{A}_C(n \text{ NS5's})$$

$\mathcal{A}_H(n\ D5\text{'s}) \equiv A_{N,n}$ admits description as quantum Hamiltonian reduction of:

$$U(\mathfrak{gl}_N) \otimes W^{Nn} \otimes U(\mathfrak{gl}_N)$$

Left D3 branes
generators $(B_-)_\alpha{}^\beta$

n fund. hypers
at the interface
Generators
$X^a{}_\alpha,\ Y_b{}^\beta \leftarrow$ Weyl algebra

Right D3 branes
generators $(B_+)_\alpha{}^\beta$

There are two homomorphisms from the Yangian $Y(\mathfrak{gl}_n)$:

- $T[z] = \mathbb{1} - X \frac{1}{z - B_+} Y$
- $T[z] = \mathbb{1} + X \frac{1}{z + B_-} Y$

In the large-N limit, $A_{N,n}$ gets "closer" to $Y(\mathfrak{gl}_n)$

$\longrightarrow$ Trace T_H on $A_{N,n}$ induces traces on $Y(\mathfrak{gl}_n)$ via these homomorphisms.

Can we make them more explicit?

Yes, on the S-dual side.

Look at $\mathcal{A}_C$(n NS5's).

Interface is engineered by a quiver:

$$\boxed{N} - N - \underbrace{N - \cdots - N - N}_{n-1 \text{ gauge nodes}} - \boxed{N}$$

$\longrightarrow$ Construct its $\mathcal{A}_C$, promote masses to fields.

The algebra itself was constructed
e.g. in [Bullimore-Dimofte-Gaiotto], [Braverman-Finkelberg-Nakajima]

To compute the trace, have to use [MD-Fan-Pufu-Yacoby]
↳ this yields representation of [Gerasimov-Kharchev-Lebedev-Oblezin] of the Yangian in terms of difference operators.

The final answer for $\langle T[z_1]_{a_1}^{b_1} T[z_2]_{a_2}^{b_2} \dots T[z_L]_{a_L}^{b_L} \rangle$
is conjecturally related to
a length-L $\mathfrak{sl}_n$ XXX spin chain

$\longrightarrow$ Twisted traces form a commutative algebra.

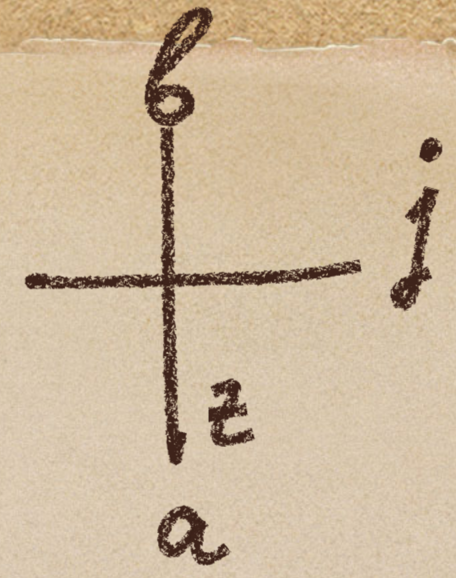
- Traces over evaluation modules (of fin.-dim. irreps of $\mathfrak{sl}_n$) can be taken as a generating set.
- Alternatively, traces over evaluation modules for Vermas.

. . .

L-operator: $\mathbb{L}(z) = z \cdot \mathbb{1} + \sum E_{ij} \otimes J_{ij}$ generators of $\mathfrak{sl}_n$; elementary $\mathfrak{sl}_n$ matrices

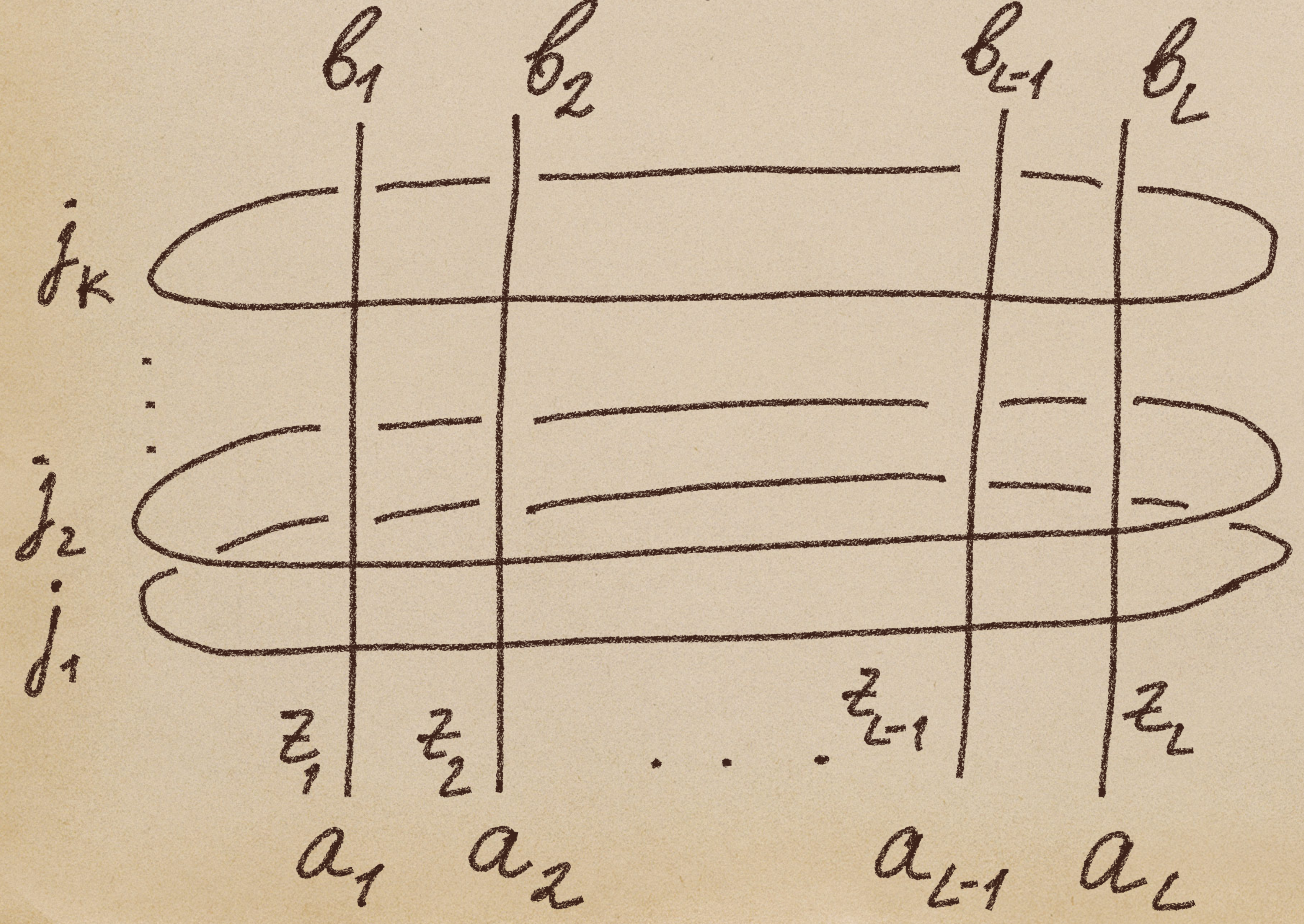
Satisfies: $R(x-y)(\mathbb{L}(x) \otimes \mathbb{1})(\mathbb{1} \otimes \mathbb{L}(y)) = (\mathbb{1} \otimes \mathbb{L}(y))(\mathbb{L}(x) \otimes \mathbb{1}) R(x-y)$

$R(z) = z + P : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$; $\mathbb{L}(z)$ determines eval. module

Pictorially: $\mathbb{L}(z)_a{}^b =$ 

, j labels sl_n rep.

Twisted trace of $T[z_1]_{a_1}{}^{b_1} \ldots T[z_L]_{a_L}{}^{b_L}$ in $[j_1] \otimes \cdots \otimes [j_K]$:



Matrix element of a product of K transfer matrices of a length-L periodic inhomogeneous sl_n spin chain.

Just need to determine representations...

So far: $\langle T[z_1]_{a_1}{}^{b_1} \ldots T[z_L]_{a_L}{}^{b_L} \rangle$

$\Downarrow$

products of transfer matrices $\mathbb{T}$
in a length-L sl_n spin chain

n = number of fivebranes

Consider $n=2$.

Using techniques of [MD-Fan-Pufu-Yacoby], can directly compute $\langle T[z_1]_{a_1}{}^{b_1} \cdots T[z_L]_{a_L}{}^{b_L} \rangle \ldots$

It turns out that...

The answer involves N sl_2 Verma modules.

I.e., $[j_1], \ldots, [j_N]$ — eval. modules corresp. to sl_2 Verma modules of h.w. j_m.

The correlator is:

$$\sum_{\sigma \in \frac{S_{2N}}{S_N \times S_N}} \frac{i^{-N(2N+1)}}{\prod_{a=1}^{N}\prod_{k=N+1}^{2N} 2\sinh\pi(\mu_{\sigma(a)}-\mu_{\sigma(k)})} \langle a_1 \dots a_L | \prod_{a=1}^{N} \mathbb{T}^{+}_{-\frac{1}{2}-\mu_{\sigma(a)}}(\vec{z}) | b_1 \dots b_L \rangle$$

Transfer matrix in eval. Verma, h.w. $= -\frac{1}{2} - \mu_{\sigma(a)}$

$$\mathbb{T}^{+}_{-\frac{1}{2}-\mu}(\vec{z}) = \frac{1}{2\sinh\pi\zeta} Q_+(\vec{z}-\mu\cdot\vec{1})\, Q_-(\vec{z}+\mu\vec{1}).$$

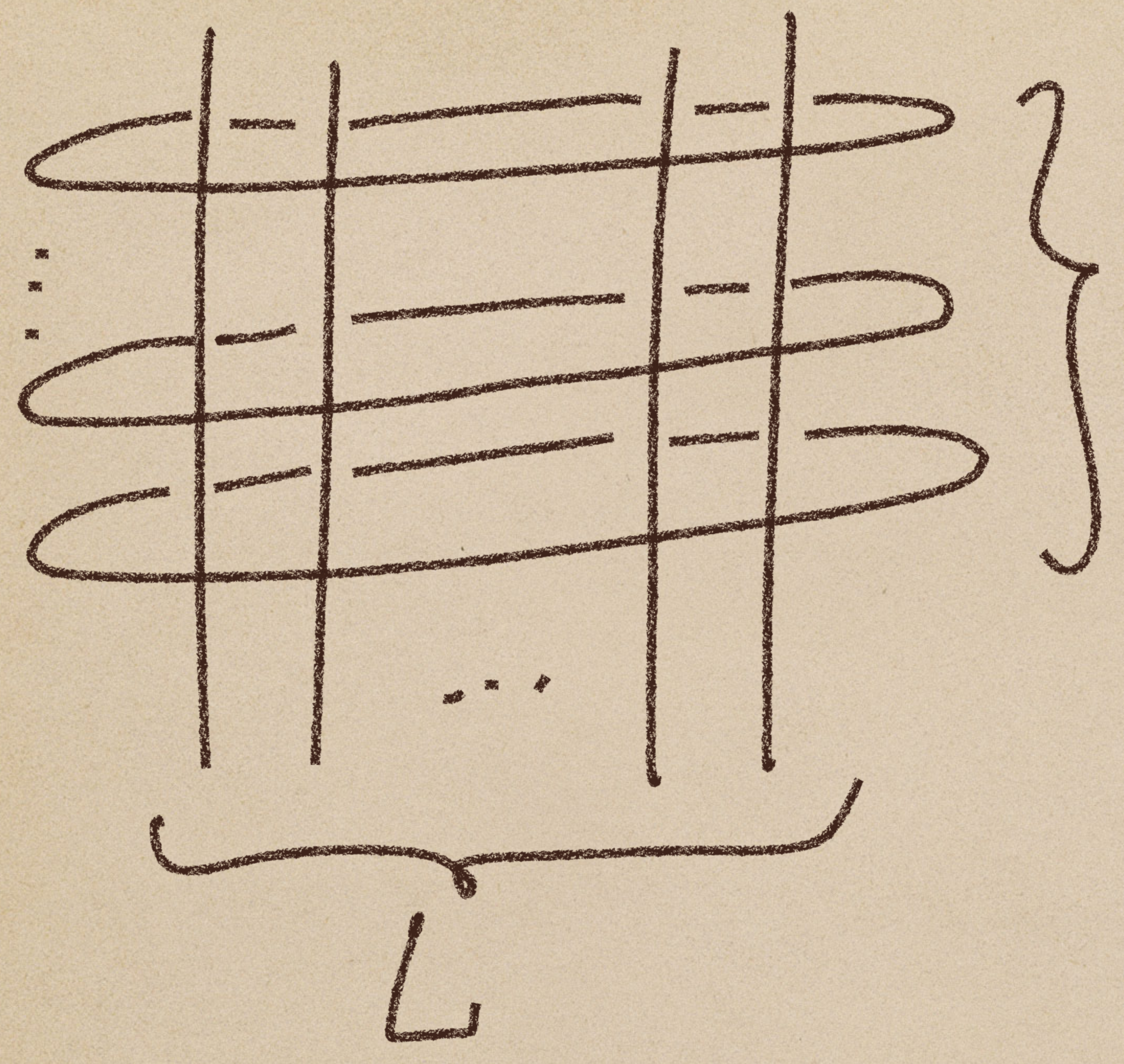
Baxter Q-operators

Remarkably explicit answer!

$Q_\pm$ can be computed algorithmically using the results of [Bazhanov-Lukowski-Meneghelli-Staudacher] (as traces over an auxiliary oscillator Fock space)

Compute $Q_\pm$ for each $L \Rightarrow$ answers for $\forall N$

Can do large-N etc...



$N \leftarrow$ the same as # of D3 branes!

To do: generalize to $n > 2$

For the nearest future: holography.

Thank You!