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Quantum K-Theory & Wilson Loop Algebras

(Correspondence b/w 3d $\mathcal{N}=2$ gauge theories & Quantum K-theory)

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Outline:

(1) Introduction

(2) Wilson Loop Algebra from Difference Equations

(3) Examples : QK & Wilson Loop Algebras

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(1) Introduction

3d $\mathcal{N}=2$ $U(M)$ gauge theory w/ N fundamentals: [Kapustin, Willet '13],...

* BPS Wilson Loops: $W_\mu = \mathrm{Tr}_\mu\left(\mathcal{P}\exp\left(i\oint A\right)\right)$ μ: rep. of $U(M)$

* Algebra: $W_\mu * W_\nu = \sum_\lambda C_{\mu\nu}^{\lambda}(Q)\, W_\lambda$

$\to$ weight of $U(1)_{top}$ (dual $U(1)_{gauge}$)
carried by vortices

[Givental '03], ...

Quantum K-theory of $Gr_M(N)$:

* K-group $K(Gr) := K^\circ(Gr_M(N)) = \langle\!\langle \mathcal{O}_\mu \rangle\!\rangle$ structure sheaves of Schubert cycles

* Quantum tensor product: $\mathcal{O}_\mu \otimes_Q \mathcal{O}_\nu = \sum \tilde{C}^{\lambda}_{\mu\nu}(Q)\, \mathcal{O}_\lambda$ g=0 GW inv. of $Gr_M(N)$

$$= \mathcal{O}_\mu \otimes \mathcal{O}_\nu + O(Q)$$

Aim: Relationship b/w $C^{\lambda}_{\mu\nu}$ & $\tilde{C}^{\lambda}_{\mu\nu}$ c.f. [Ueda, Yoshida '19], [Gu, Mihalcea, Sharpe, Zou '20]

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(2) Wilson Loop Algebra from Diff. Equations

Chern-Simons terms: $U(M) = (U(1) \times SU(M))/\mathbb{Z}_M$, $U(1)_R$-sym.

$$S_{CS} = \frac{\kappa}{4\pi}\int \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A^3\right)$$

Lie algebra valued connection 1-form

CS levels: $\kappa = (\kappa_A, \kappa_S, \kappa_R)$

$U(1)_{gauge}$ $SU(M)_{gauge}$ mixed $U(1)_{gauge} \times U(1)_R$

$S^1 \times_q D^2$ - Partition function:

Berry conn. (Cher. class) effect. CS levels

$$Z_{S^1\times_q D^2}(q,Q) = \operatorname*{Res}_{\vec{\epsilon}=0} f(q,\epsilon_1,\dots,\epsilon_M)\, I^{\hat{\kappa}}(q,Q,\epsilon_1,\dots,\epsilon_M)$$

3d Gauge Theory / QK Correspondence:

ϵ_a : Chern roots of taut. subbundle S^* of $Gr_M(N)$

level structure (for $\hat{\kappa} \neq 0$) [Ruan, Zhang '18]

$$\Rightarrow I^{\hat{\kappa}}(q,Q;\epsilon_1,\dots,\epsilon_M) \cong I^{K,\hat{\kappa}}_{Gr_M(N)}(q,Q) \in \mathcal{K}$$ c.f., [Wen '19]

(perm. sym.) K-theo. Givental

I-function of $Gr_M(N)$ [Givental '15], ...

w/ $\mathcal{X} = \mathcal{X}_+ \oplus \mathcal{X}_-$: $\mathcal{X}_+ = K(Gr) \otimes \mathbb{C}[q^{-1}, q] \otimes \mathbb{C}[[Q]]$

$\mathcal{X}_- = K(Gr) \otimes \{ r(q) \in R(q) \mid r(0) \neq \infty, r(\infty) = 0 \} \otimes \mathbb{C}[[Q]]$

(param. sym.) input; Correlators ($g=0$ GW inv.)

$$\Rightarrow I^{k,\hat{x}}_{Gr_M(N)} = (1-q) + \underbrace{t}_{\in \mathcal{X}_+} + \underbrace{I_{Cor}(t)}_{\in \mathcal{X}_-}$$

for $\frac{1}{M} \left| x_A + (M-1) x_S \right| \leq \frac{N}{2}$: $t=0$ (ordinary) QK I-function

Higgs branch $Gr_M(N)$

c.f. $\mathbb{P}^{N-1}$: [Intriligator, Seiberg '13]

Wilson Lines:

Abelianization $U(1)^M \subset U(M)$ max. torus $\Rightarrow Z^{Ab}_{S^1 \times D^2}(q, Q_1, \ldots, Q_M)$

BPS Wilson line @ $S^1 \times \{0\} \subset S^1 \times D^2$ w/ charge $\vec{n} = -(n_1, \ldots, n_M)$ w.r.t $U(1)^M$

$$\langle W_{\vec{n}} \rangle_{S^1 \times D^2} = \mathcal{D}_{\vec{n}} Z^{Ab}_{S^1 \times D^2} \quad \text{w/ } \mathcal{D}_{\vec{n}} = \prod_{a=1}^{M} q^{n_a \theta_a} \; ; \quad \theta_a = Q_a \frac{\partial}{\partial Q_a}$$

$$\langle \hat{W}_{\vec{n}} \rangle_{S^1 \times D^2} = \hat{\mathcal{D}}_{\vec{n}} Z^{Ab}_{S^1 \times D^2} \quad \text{w/ } \hat{\mathcal{D}}_{\vec{n}} = \prod_{a=1}^{N} \delta_a^{n_a} \; ; \quad \delta_a = 1 - q^{\theta_a}$$

Algebra of diff. op.: $\hat{\mathcal{D}}_{\vec{n}} \cdot \hat{\mathcal{D}}_{\vec{m}} = \hat{\mathcal{D}}_{\vec{n}+\vec{m}}$

+ relations: $\mathcal{L}_a \hat{I}^{k,\hat{x}}_{Ab} = 0$ w/ $\mathcal{L}_a = \delta_a^N + Q_a P_a(\hat{x}, q, \delta_a)$

$U(M)$ gauge theory: $\hat{\mathcal{D}}_\mu \cdot \hat{\mathcal{D}}_\nu = \sum_{\lambda \in B_{Gr_M(N)}} C^\lambda_{\mu\nu}(Q,q) \hat{\mathcal{D}}_\lambda$; $Q_a \equiv Q$

w/ $\hat{\mathcal{D}}_\mu = \sigma_\mu(\delta_1, \ldots, \delta_M)$

↳ Schur poly. ↳ Young tableaus of $Gr_M(N)$ Schubert cycles

Algebra of Wilson lines: $\mathcal{L}_a$ dep. on Q_a $\Rightarrow$ δ_a action on relations

BUT: $\delta_a\left(f(Q)\, I(Q_a)\right) = f(Q_a)\left(\delta_a I(Q_a)\right) + O(1-q)$

Wilson line algebra: $\boxed{\hat{W}_\mu * \hat{W}_\nu = \sum_\lambda C^{\lambda}_{\mu\nu}(Q, q=1)\, \hat{W}_\lambda}$

(3) Examples

(i) factorized case: $P_a(\hat{\kappa}, q, Q_a) \equiv 1$, cond. on $\hat{\kappa}$

$\mathcal{L}_a = \delta_a^N + Q_a$, $a = 1, \ldots, M$: Relations of QCoh. of $Gr_M(N)$

[Vafa '91; Intriligator '91; Siebert, Tian '94; ...]

Wilson line alg. $\cong$ QCoh. of $Gr_M(N)$

$\cong$ Verlinde alg. of gauged WZW $\mathcal{U}(M)/\mathcal{U}(M)$, level $N-M$

[Witten '93]

(ii) Canonical CS level : $\hat{\kappa} \equiv 0$

* QK theory of $Gr_M(N)$ w/o level structure [Buch, Mihalcea '11]

* Buch, Mihalcea: Grothendieck Polyn. $\mathcal{O}_\mu(1-e^{-\sigma_a})$

$\left(\mathcal{O}_\mu = M_\mu^{\nu} \delta_\nu \text{ linear trans. } M \text{ w/ det.} 1\right)$

(iii) 2d limit: $\sigma_\mu(\delta_a) \xrightarrow{2d} \sigma_\mu(\sigma_a)$, $Q^{3d} \to e^{-N \log \beta} Q_a$

$\mathcal{L}_a \to \sigma_a^N + Q_a^{2d}$ (ind. of $\hat{\kappa}$)

$\Rightarrow$ QCoh algebra for any $\hat{\kappa}$