Holographic 3-Point Functions & Their Applications (Real-Time, Finite-Temperature)

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- **Real-time Finite-temperature Holographic 3-Point Functions**

- **Application (I) : 2\textsuperscript{nd} Order Conformal Relativistic Hydrodynamics**

- **Application (II) : Jet Quenching with Finite Chemical Potential**
  Barnes, Vaman & CW : Work in process
A Brief Review of Holography

- **Quark-Gluon Plasma at RHIC & LHC**
  - Strong-interacting: Strongly-coupled QCD
  - Nearly-perfect fluid: Hydrodynamics

- **Holography (Gravity/Gauge Duality, AdS/CFT duality…)**
  - Feynman diagram method breaks down at strong coupling
  - Alternative approach to compute strongly-coupled QFT
  - Not known exactly for QCD
  - Simplest example:
    - $\text{AdS}_5$ (IIB Supergravity, weakly-coupled) / $\text{CFT}_4$ ($\mathcal{N}=4$ SYM, strongly-coupled)
    - Extended to non-conformal FTs

- **Real-Time & Finite-Temperature**
  - Finite-temperature: AdS + Black Holes (Schwarzschild, RN…)
  - Real-time: various types of correlators (causal, time-ordered, non-time-ordered…)
    - Skenderis & van Rees (arXiv:0805.0150 & 0812.2909)

- **2-pt correlators were well studied (e.g. $\eta$, $\sigma$…, 2002-)**
  What higher n-pt correlators can do? 3-pt? (e.g. $\tau$, $\lambda_i$…, not known )
2nd Order Conformal Relativistic Hydrodynamics – Review

• Early literature:
  – Baier, Romatschke, Son, Starinets & Stephanov, arXiv:0712.2451
  – Bhattacharyya, Hubeny, Minwalla & Rangamani, arXiv:0712.2456

• Our approach: inspired by

\[ \eta = \lim_{\omega, k \to 0} i \partial_{\omega} < T^{xy} T^{xy}(\omega, k) >_{ra} \]

2nd order hydro coefficients = \( \lim \) 3-point functions \( <TTT> \)

• Ingredient (1): **Kubo formulae for 2nd order hydro coefficients**

• Ingredient (2): **Holographic 3-pt functions**
  – Arnold, Barnes, Vaman & CW, arXiv:1004.1179

\[ G_{aar}^{(3)} \sim \delta^{(4)}(p_1 + p_2 + p_3) \int_0^1 du \sqrt{-g} \Delta_A(p_1, u) \Delta_A(p_2, u) \Delta_R(p_3, u) \]
Deriving 2nd order Kubo formulae  

(Moore & Sohrabi, arXiv:1007.5333)

- **Hydrodynamic expansion** of energy-stress tensor:
  
  \[ T^{\mu\nu}[h] = (\varepsilon + P)u^\mu u^\nu + Pg^{\mu\nu} - \eta \sigma^{\mu\nu} + \eta \tau_\Pi \left( \langle u \cdot \nabla \sigma^{\mu\nu} \rangle + \frac{\nabla \cdot u}{3} \sigma^{\mu\nu} \right) + \kappa \left( R^{<\mu\nu>} - 2u_\alpha u_\beta R^{\alpha<\mu\nu>\beta} \right) + \lambda_1 \sigma^{<\mu\nu>\rho} + \lambda_2 \sigma^{<\mu\nu} \Omega^{\nu>\rho} + \lambda_3 \Omega^{<\mu\nu} \Omega^{\nu>\rho} \]

- **EOMs & Constraints** (on flat background):
  
  \[
  \begin{align*}
  \nabla_\mu T^{\mu\nu} &= 0 \\
  T^{\mu}_{\mu} &= 0 & \Rightarrow & \quad \varepsilon(x) \\
  u^2 &= -1 & \Rightarrow & \quad P(x) \\
  \end{align*}
  \]

- **Solve**: double expansion of \( h_{\mu\nu} & (\omega, k) \)
  
  \[
  \varepsilon = \overline{\varepsilon} + \frac{4\overline{\varepsilon}}{3\omega^2 - k^2} \left( -\omega kh_{iz} - \frac{1}{2} k^2 h_{it} - \frac{1}{2} \omega^2 h_{ii} \right) + O\left( h^2, \omega, k \right)
  \]

- **Compare** with correlator expansion
  
  \[
  T^{\mu\nu}[h] = G^{\mu\nu}_{r}(x) - \frac{1}{2} \int G^{\mu\nu,\alpha\beta}_{ra}(x,y)h_{\alpha\beta}(y)d^4y + \frac{1}{8} \int G^{\mu\nu,\alpha\beta,\gamma\delta}_{raa}(x,y,z)h_{\alpha\beta}(y)h_{\gamma\delta}(z)d^4yd^4z
  \]
Hydrodynamic expansion of correlators:

\[
G^{xz,tx,xx}_{aar}(1,2,3) \bigg|_{k_i=0, \omega_2=0} = -i2\pi T_\eta \frac{\omega_1^2 k_2}{3\omega_1^2 - k_2^2} + 2\pi^2 T^2 \frac{\omega_1 k_2 \left( \lambda_2 (-9\omega_1^4 + 9\omega_1^2 k_2^2 - 2 k_2^4) + \kappa \cdots + \eta_\Pi \cdots + \eta^2 / \bar{\epsilon} \cdots \right)}{(3\omega_1^2 - k_2^2)^2}
\]

Kubo formula: **zoom into** a particular term by taking a certain limit:

Order of limits matters sometimes due to the pole:

\[
\begin{align*}
\left. \lim_{k_2 \to 0} \frac{\partial}{\partial k_2} \left. \lim_{\omega_1 \to 0} \frac{\partial}{\partial \omega_1} G^{xz,tx,xx}_{aar}(1,2,3) \right|_{k_i=0, \omega_2=0} = & -\lambda_2 - \kappa + 2\eta_\Pi \\
\left. \lim_{\omega_1 \to 0} \frac{\partial}{\partial \omega_1} \left. \lim_{k_2 \to 0} \frac{\partial}{\partial k_2} G^{xz,tx,xx}_{aar}(1,2,3) \right|_{k_i=0, \omega_2=0} = & -\frac{1}{2} \lambda_2 - \frac{1}{2} \kappa + \frac{4}{3} \eta_\Pi + \frac{\eta^2}{2\bar{\epsilon}}
\end{align*}
\]

Can compare the **whole** 3-point functions from AdS with the hydrodynamic expansions **term by term** (fit into the templates)

→ A good consistency check on AdS calculation too!
2nd Order Conformal Relativistic Hydrodynamics – AdS Side

• Energy-stress tensor $T^{\mu\nu} \leftrightarrow$ metric fluctuation $h_{\mu\nu}$

• **Solve** linearized **Einstein equation** on AdS-Schwarzschild BH background small $\omega, k$ to 2nd order $\rightarrow$ Boundary-to-bulk propagators

  (Policastro, Son & Starinets, hep-th/0205052 & 0210220)

• Perturbatively **expand** Einstein-Hilbert & Gibbons-Hawking **action** to $O(h^3)$ to get cubic vertices

  (Arutyunov & Frolov, hep-th/9901121, need to restore boundary terms)

• **Do** the bulk **integral**

$$G^{\alpha\beta, \gamma\delta, \mu\nu}_{aar}(1, 2, 3) \sim \delta^{(4)}(p_1 + p_2 + p_3) \int_0^1 du \sqrt{-g}V^{\alpha\beta, \gamma\delta, \mu\nu}(u)h_{A,\alpha\beta}(p_1, u)h_{A,\gamma\delta}(p_2, u)h_{R,\mu\nu}(p_3, u)$$

Gauge Condition:

$$h_{\mu 5} = 0$$
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- **Divergent at both boundaries !!!**
  - Boundary divergence (at $u=0$):
    Add gravitational counter terms: volume + curvature + …

  *(Balasubramanian et al, hep-th/9902121, 9903238, 9906127 & 0002230 et al)*

  Completely removed 😊
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• **Do** the bulk **integral**

  $$G_{\alpha\beta,\gamma\delta,\mu\nu}^{(1,2,3)} \sim \delta^{(4)}(p_1 + p_2 + p_3) \int_0^1 du \sqrt{-g} V_{\alpha\beta,\gamma\delta,\mu\nu}^{(A)}(u) h_{A,\alpha\beta}^{(p_1,u)} h_{A,\gamma\delta}^{(p_2,u)} h_{R,\mu\nu}^{(p_3,u)}$$

• **Divergent at both boundaries !!!**
  - Horizon divergence (at $u=1$):
    - Due to **gauge** freedom (purely-gauge solutions near horizon)
    - Form gauge-invariant combinations or limits

    Completely removed 😊
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• **Do** the bulk **integral**

  $$G_{\alpha\beta,\gamma\delta,\mu\nu}^{(1,2,3)} \sim \delta^{(4)}(p_1 + p_2 + p_3) \int_0^1 d\varpi \sqrt{-g} V_{\alpha\beta,\gamma\delta,\mu\nu}(\varpi) h_{A,\alpha\beta}(p_1, \varpi) h_{A,\gamma\delta}(p_2, \varpi) h_{R,\mu\nu}(p_3, \varpi)$$

• **Finite results** 😊
Results: (Arnold, Vaman, CW & Xiao, arXiv:1105.4645)

- **Gauge-invariant limit:**

  \[
  G_{raa}^{xy,tx,ty} (k_3 = -k_1 - k_2; k_1, k_2) \bigg|_{\omega_1, \omega_2 = 0} = -P + \kappa^2 T^2 \left( 2\kappa (k_1^2 + k_2^2) - \lambda_3 k_1 k_2 \right) + O(k^4) \\
  = -\frac{\pi^2 N^2 T^4}{8} + \frac{\pi^2 N^2 T^4}{4} (k_1^2 + k_2^2) + O(k^4)
  \]

  ... hydro

  ... AdS

- **Gauge-invariant combination:**

  \[
  \left( G_{raa}^{xx,tx,tx} + G_{raa}^{yy,tx,tx} - 2(1-k_3^2)G_{raa}^{zz,tx,tx} + \frac{2}{3} k_3^2 G_{raa}^{tt,tx,tx} - 2ik_3 G_{raa}^{tz,tx,tx} \right) \bigg|_{\omega_1, \omega_2 = 0} = -2P + \left( \frac{4}{3} P - 4\pi^2 T^2 \kappa \right) (k_1^2 + k_2^2) + \left( \frac{8}{3} P + 2\pi^2 T^2 \lambda_3 \right) k_1 k_2 + O(k^4) \\
  = -\frac{\pi^2 N^2 T^4}{4} - \frac{\pi^2 N^2 T^4}{3} (k_1^2 + k_2^2 - k_1 k_2) + O(k^4)
  \]

  ... hydro

  ... AdS

- **Consistent results:**

  \[
  \lambda_1 = \frac{1}{16} N^2 T^2, \quad \lambda_2 = -\frac{\ln 2}{8} N^2 T^2, \quad \lambda_3 = 0, \quad \kappa = \frac{1}{8} N^2 T^2, \quad \tau_\Pi = \frac{2 - \ln 2}{2\pi T}
  \]
1) All 2nd order hydrodynamic coefficients (for conformal QGP) were derived via Kubo formulae (related to causal 3-pt correlators).

2) Prescriptions to compute real-time higher n-pt correlators via AdS/CFT had been built up.

3) Causal 3-pt correlators in (1) were computed via (2), thus all 2nd order hydrodynamic coefficients were obtained.
   – Extend & complete Baier et al’s approach;
   – An independent check to Bhattacharyya et al’s results.

4) Can be generalized to include higher-derivative corrections, to non-conformal cases…
Thanks!