Boundary Conflicts and Cluster Coarsening: Waves of Life and Death in the Cyclic Competition of Four Species

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Friday, October 21
Motivation

Examples

- 3 Lizard populations competing cyclically
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- 5 grass populations competing in a complicated manner
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- 4 species cyclically competing model is a stepping stone in understanding complex food chains of 4 species.
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Model

Definition

On a two dimensional square lattice with periodic boundary conditions and an occupation number of 1 per lattice site we randomly distribute the four species $A, B, C$ and $D$ which compete in the following manner:

- $A + B \xrightarrow{k} A + A$
- $A + B \xrightarrow{1-k} B + A$
- $B + C \xrightarrow{k} B + B$
- $B + C \xrightarrow{1-k} C + B$
- $C + D \xrightarrow{k} C + C$
- $C + D \xrightarrow{1-k} D + C$
- $D + A \xrightarrow{k} D + D$
- $D + A \xrightarrow{1-k} A + D$
- $A + C \xrightarrow{\mu} C + A$
- $B + D \xrightarrow{\mu} D + B$
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\[
\begin{align*}
A + B & \overset{k_A}{\longrightarrow} A + A \\
B + C & \overset{k_B}{\longrightarrow} B + B \\
C + D & \overset{k_C}{\longrightarrow} C + C \\
D + A & \overset{k_D}{\longrightarrow} D + D \\
A + C & \overset{\mu}{\longrightarrow} C + A \\
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\(\mu A + C \xrightarrow{\mu} C + A\) and \(B + D \xrightarrow{\mu} D + B\)
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- $D + A \xrightarrow{k_D} D + D$ else $D + A \xrightarrow{1-k_D} A + D$
- $A + C \xrightarrow{\mu_{AC}} C + A$ and $B + D \xrightarrow{\mu_{BD}} D + B$
Model Cont’d

Figure: Periodic Boundary
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Definition

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The sum of the populations of the species is invariant and is equal to $L^2$ since the occupation number is 1 element per lattice site.
Three Species and Pattern Formation

Figure: Pattern Formation in 3 Species Model [E. Frey Group]
Remark

- Alliance Formation \((A, C)\) vs. \((B, D)\)
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Figure: $t \sim 100$
Four Species and Cluster Coarsening

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Figure: \(t \sim 100\)

Figure: \(t \sim 500\)
Space Time Correlation function

**Definition**

\[
\lambda^A_t(i, j) = \begin{cases} 
1 & \text{if } (i, j) \text{ contains } A \text{ at time } t \\
-1 & \text{Otherwise.}
\end{cases}
\]

\[
\gamma^A_t(i, j, r) = \frac{1}{4} \lambda^A_t(i, j)[\lambda^A_t(i + r, j) + \lambda^A_t(i - r, j) + \lambda^A_t(i, j + r) + \lambda^A_t(i, j - r)]
\]

we similarly define \(\gamma^\xi_t(i, j, r)\) and \(\lambda^\xi_t(i, j)\) where \(\xi = B, C \text{ or } D\). Then the space-time correlation function is defined as

\[
C_t(r) = \frac{L}{4} \sum_{r=1}^{L} \sum_{i=1}^{L} \sum_{j=1}^{L} [\gamma^A_t(i, j, r) + \gamma^B_t(i, j, r) + \gamma^C_t(i, j, r) + \gamma^D_t(i, j, r)]
\]
Measurement

We plot \( \frac{1}{r} C_0(0) = \text{constant} \) where \( r \in \mathbb{R} \) chosen appropriately as well as \( C_t(r) \) as function of time, then we plot the intersection of the two traces as a function of time yielding the length scale \( L \sim t^{1/z} \) as a function of time.

Problem

Are there "better" geometries to observe the length scale?
Correlation Function and Geometry

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- Can we observe periodic oscillations in the population sizes as a function of time for the symmetric reaction rates like those observed in the zero dimensional model?
- Which confined geometry provides the largest time regime where the dynamical exponent can be observed free of early time effects or finite-size effects? (Work In Progress.)
Questions

Do you have a question? Ask Away…