An Averaged-Separation of Variables Solution to the Burger Equation

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then we have a separation of variables and both X(x) and T(t) can, in principle, be calculated.

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- However, almost all of these special solutions have some physical application if the original PDE models a given physical phenomenon.

Major Goal

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Investigate the possibility of combining the SOV methods with an "averaging" technique... to see if this methodology can be used to generilize the SOV procedure.

Comments

- SOV generated solutions do not, in general, give valid solutions of initial-value and/or boundary-value problem.
- Our test equation is the Burger's equation

$$\mathrm{u}_t + uu_x = 0, \qquad u = u(x,t) \qquad u(x,0) \qquad given$$

Exact Solution

$$\mathrm{u}_t + uu_x = 0, \mathrm{u(x,0)} \text{ given},$$

$\therefore$

$$\mathrm{u(x,t)=u(x\text{-}ut,o)}$$

Proof: Substitute the above expression in the PDE and carry out the required differentiations. This is done in (almost) all introductory text books on PDE's.

Let

$$u(x,0)=x(1-x), \qquad 0\leq x\leq 1.$$

$$u(0,t)=0,\ u(1,t)=0.$$

$\therefore$, exact solution is

$$u=(x-ut)-(x-ut)^2$$

and the bounded, non-negative solution is

$$u(x,t)=\left(\frac{1}{2t^2}\right)\left[-(1-2xt+t)+\sqrt{(1-2xt+t)^2+4t^2x(1-x)}\right].$$

Note

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- u(0,t)=u(1,t)=0
- For x fixed in $0 < x < 1$, then for large t,

$$u(x,t) \sim \left(\frac{1}{2t^2}\right)\left[(2xt - t) + \sqrt{(2xt - t)^2 + 4t^2 x(1 - x)}\right].$$

$$u(x,t) \sim \frac{x}{2t}.$$

Exact SOV Solution

$u_t + uu_x = 0$

- Assume u(x,t)=X(x)T(t)
- $\therefore \quad u(x,t) = \frac{Ax+B}{At+1}, \qquad$ (A,B) arbitrary constants.

$$u(x,t) \xrightarrow[\substack{\text{t- large} \\ x-fixed}]{} \frac{x+\frac{B}{A}}{t}.$$

- Since for this solution, u(x,0)=Ax+B, this implies that

  u(x,0)=x(1-x) does not correspond to any SOV solution.

An Averaging Based Solution

- Assume u(x,t)=x(1-x) and u(x,t)=x(1-x)T(t).

- Substitution into the PDE gives

$$(\text{x-x}^2)T' + (x-x^2)(1-2x)T^2 = 0$$

$$\left\langle (x-x^2) \right\rangle = \tfrac{1}{6}, \quad \left\langle (x-x^2)(1-2x) \right\rangle = 0,$$

where

$$\langle (\ldots) \rangle \equiv \int_0^1 (\ldots) dx$$

- $\therefore \qquad \mathrm{T}' = 0, \quad T(0) = 1$

$\therefore$ Averaged equation gives $\mathrm{T}'(t) = 0 \qquad T(t) = T_0 = constant.$
Averaged solution is then

$$\mathrm{u}_a(x,t) = x(1-x).$$

The solution, in this approximation is independent of time!

$$\mathrm{u}_a(x,0) = x(1-x) \qquad \Longrightarrow u_a(x,t) = x(1-x).$$

$$T(t) = \tfrac{1}{1+2t}.$$

Averaged/SOV Solution

$$u_t + uu_x = 0, \qquad u(0,t) = 0, \qquad u(1,t) = 0.$$

Assume $u(x,0)=(x-x^2)[1 + 0(x^2)]$
and take $u_a(x,t) = xf_1(t) - x^2 f_2(t)$,
with
$f_1(0) = f_2(0) = 1, \qquad u_a(x,0) = x(1-x)$.

Differentiating, we have
$u_t = xf_1' - x^2 f_2', \qquad u_x = f_1 - 2xf_2. \qquad 0(x^2)$.

Putting these back in the PDE and collecting terms gives

$x(f_1' + f_1^2) - x^2(f_2' + 3f_1 f_2) + x^3(2f_2^2) \doteq 0$

Requiring coefficients of x and x^2 to be zero gives

$$f_1' + f_1^2 = 0$$

Integrating and using the initial condition $f_1(0) = 1$ gives

$$f_1(t) = \frac{1}{1+t}.$$

Similarly,

$$f_2' = -3f_1 f_2 = -\frac{3}{1+t} f_2.$$

Integration gives

$\ln(f_2(1+t)^3) = c$, a constant.

When t=0, $\ln(f_2(1+t)^3) = 0$
or

$$\mathrm{f}_2(t) = \frac{1}{(1+t)^3}.$$

Hence

$$\mathrm{u}_a(x,t) = \frac{x}{1+t} - \frac{x^2}{(1+t)^3}.$$

$$u_{exact}(x,t) \xrightarrow[\substack{\text{t- large} \\ x-fixed}]{} \frac{x}{t}.$$

Conclusions

- Both the approximate and exact solutions satisfy the initial and boundary-value conditions, and the condition of positivity.

- The extension of the assumed averaged solution has many features in common with the exact solution.

- **General Conclusion:** Space averaging techniques may give qualitative correct results for approximations to the solution of the PDE's, but the qualitative details mayl not be accurate. The generalization of this method, where the coefficients depend on time may be a valuable new procedure to study.