Qualitative Analysis of a Differential Equation of Abel

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Goals

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- Study the properties of the solutions of a differential equation of Abel type...
- Calculate an analytic approximation to the periodic solutions of this ODE..

$$dy/dt = sint - y^3$$

Reference: Uri Elias, Vol. 115, MAA Monthly (Feb. 2008) 147-149

Physical System Modeled

Physical System Model

- Periodic forced, nonlinear oscillator

$$\varepsilon d^2y/dt^2 + dy/dt + y^3 = sint$$

- Massless limiting case ($\varepsilon -> 0$)

$$\underline{dy/dt = sint - y^3}$$

The general form of an Abel equation of the first-kind is

$$dy/dt = f_0(t) + f_1(t)y + f_2(t)y^2 + f_3(t)y^3.$$

Special Case

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- Unforced case

$$dy/dt = -y^3,\; y(0) = y_0$$

- Exact solution

$$y(t) = \frac{y_0}{\sqrt{1 + 2y_0^2 t}}$$

- Solution has a singularity at $t^* = -\left(\frac{1}{2y_0^2}\right)$.
- For given y_0 , the solution is defined on the interval $t^* < t < \infty$.
- $y(t) -> 0$ as $t -> \infty$.

Special case

$$y' = sint - y^3$$

- For sufficiently negative y, we can always make
  $$|y^3| \gg 1 \geq |sint|$$
  and for this range of y values the ODE is (to a very good approximation)
  $$y' = -y^3.$$
- Conclusion: For sufficiently large t,
  $$Max|y(t)| < 1.$$

Comment

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- The qualitative theory of ODE's can be used to not only reach the previous conclusions, but to also show that if $y_0 > 0$, then all solutions oscillate.
- Further study shows that the ODE has a unige periodic solution to which all other solutions are attracted.

Reference R.E. Mickens, Mathematical Methods for the Natural and Engineering Sciences (World Scientific, London, 2004).
(2-Dim phase-space analysis-geometrical techniques based on the existence of null-clines.)

Approximation to Periodic Solution

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(**Harmonic Balance Method**)

- Assume

$$y(t) = Asint + Bcost$$

as a first-approximation to the periodic solution.

- Substitution of this into the ODE and setting the coefficients of (sint) and (cost) of the resulting exoression to zero gives

$$A = -\left(\frac{3B}{4}\right)\left(A^2 + B^2\right)$$

$$-B = 1 - \left(\frac{3B}{4}\right)\left(A^2 + B^2\right)$$

- <u>Solution:</u> A= 0.427, B=-0.756

Results

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- $y(t) \approx (0.427)sint - (0.756)cost$
- $Max|y(t)| \approx 0.868$

Qualitative AnalysisFig.pdf

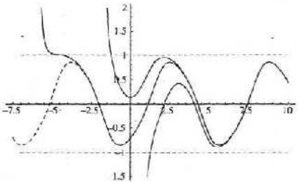

Figure 1. Four solutions, the periodic one is dashed.

Summary

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- The qualitative methodology combined with the method of harmonic balance is very powerful.
- An excellent approximation was calculated for the periodic solution.
- The periodic solution is an attractor.
- Future Problem: Construct higher-order approximations to the periodic solution.