Qualitative Analysis of a Differential Equation of Abel

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Goals

- Study the properties of the solutions of a differential equation of Abel type...
- Calculate an analytic approximation to the periodic solutions of this ODE..

\[
\frac{dy}{dt} = \sin t - y^3
\]

Physical System Model

- Periodic forced, nonlinear oscillator

\[ \varepsilon d^2 y/dt^2 + dy/dt + y^3 = \sin t \]

- Massless limiting case (\(\varepsilon - > 0\))

\[ dy/dt = \sin t - y^3 \]

The general form of an Abel equation of the first-kind is

\[ dy/dt = f_0(t) + f_1(t)y + f_2(t)y^2 + f_3(t)y^3. \]
Special Case

- Unforced case
  
  \[ \frac{dy}{dt} = -y^3, \quad y(0) = y_0 \]

- Exact solution

  \[ y(t) = \frac{y_0}{\sqrt{1 + 2y_0^2t}} \]

- Solution has a singularity at \( t^* = -\left(\frac{1}{2y_0^2}\right) \).

- For given \( y_0 \), the solution is defined on the interval \( t^* < t < \infty \).

- \( y(t) \rightarrow 0 \) as \( t \rightarrow \infty \).
y' = sint − y^3

- For sufficiently negative y, we can always make
  \[ |y^3| \gg 1 \geq |\text{sint}| \]
  and for this range of y values the ODE is (to a very good approximation)
  \[ y' = -y^3. \]

- **Conclusion:** For sufficiently large t,
  \[ Max|y(t)| < 1. \]
Comment

- The qualitative theory of ODE’s can be used to not only reach the previous conclusions, but to also show that if $y_0 > 0$, then all solutions oscillate.

- Further study shows that the ODE has a unige periodic solution to which all other solutions are attracted.

(2-Dim phase-space analysis-geometrical techniques based on the existence of null-clines.)
Approximation to Periodic Solution

(Harmonic Balance Method)

- Assume

\[ y(t) = As\int + B\cos t \]

as a first-approximation to the periodic solution.

- Substitution of this into the ODE and setting the coefficients of (\(s\int\)) and (\(\cos t\)) of the resulting expression to zero gives

\[ A = -\left(\frac{3B}{4}\right) (A^2 + B^2) \]

\[ -B = 1 - \left(\frac{3B}{4}\right) (A^2 + B^2) \]

- Solution: \( A = 0.427 \), \( B = -0.756 \)
Results

- $y(t) \approx (0.427) \sin t - (0.756) \cos t$
- $\max |y(t)| \approx 0.868$

Qualitative AnalysisFig.pdf
Summary

- The qualitative methodology combined with the method of harmonic balance is very powerful.
- An excellent approximation was calculated for the periodic solution.
- The periodic solution is an attractor.
- **Future Problem:** Construct higher-order approximations to the periodic solution.