Unusual criticality in a generalized 2D XY model

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Vortices as a topological exitation in superfluid

Superfluid order parameter: $\psi(\vec{r},t) = e^{i\theta(\vec{r},t)}$
Superfluid velocity: $\vec{V}_s(\vec{r},t) = \frac{\hbar}{m}\nabla\theta(\vec{r},t)$

since wavefunction is single valued
phase circulation(change of phase over closed path):
$\oint \nabla\theta \cdot d\vec{l} = 2\pi n \qquad n = 0, \pm 1, \pm 2, ...$

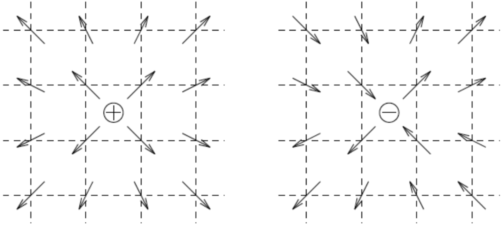


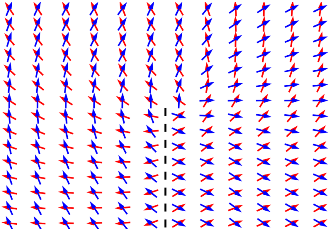
Figure: Vortices with charge ± 1

Not only integer vortices!

In a spinor-1 Bose condensate,

$$H_{int} = c(\phi^\dagger \mathbf{S} \phi)^2$$

$\phi = \mathbf{a} + i\mathbf{b}$ and $(S_i)_{jk} = -i\epsilon_{ijk}$
So, $\phi^\dagger \mathbf{S} \phi = 2\mathbf{a} \times \mathbf{b}$
If $c < 0$, $\mathbf{a}$ and $\mathbf{b}$ are parallel, and $\phi = \mathbf{n}e^{i\theta}$
Notice that $(\mathbf{n}, \theta) = (-\mathbf{n}, \theta + \pi)$, which enables half vortices.



Kosterlitz-Thouless transition and vortices

Consider in 2D, the energy and entropy of a integer vortex,

$$E = \frac{n_s}{m}\int \mathbf{v}^2 = \frac{\pi n_s \hbar^2}{m} lnL$$

$$S = k_B ln(L)^2$$

Free energy is:

$$F = U - TS = (\frac{\pi n_s \hbar^2}{m} - 2k_B T) lnL$$

So free vortices emerge when $T_v > \frac{\pi n_s \hbar^2}{2m}$

For half vortices, since the energy is 4 times smaller, the transition temperature is also 4 times smaller,

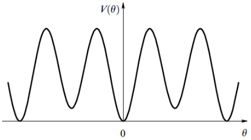
$$T_{hv} = T_v/4$$

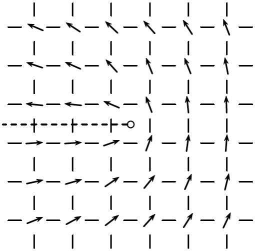
Hamiltonian

Add a term that has π periodicity to the XY Hamiltonian.

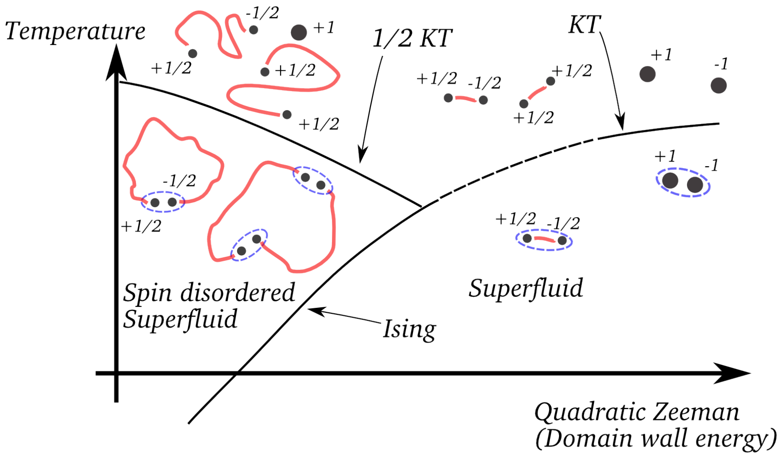
$$H = \sum_{<i,j>} [(1-\Delta)cos(\theta_i - \theta_j) + \Delta cos(2(\theta_i - \theta_j))]$$

- The extra term enables half vortices!
- Domain walls here are similar to Ising model
- Try to understand this Universality class





Schematic phase diagram



Villain Model

Use Villain model,
Partition function is changed to Gaussian form

$$Z = \int_{-\pi}^{\pi} \prod_{c} \frac{d\theta_c}{2\pi} \prod_{<ab>} \omega(\theta_a - \theta_b)$$

where $\omega(\theta) = \omega_V(\theta) + e^{-K}\omega_V(\theta - \pi)$,
and $\omega_V(\theta) = \sum_{p=-\infty}^{+\infty} e^{-\frac{J}{2}(\theta+2\pi p)^2}$ which is the Villain weight.

$$\omega(\theta) = \sum_{p=-\infty}^{+\infty} e^{-\frac{J}{2}(\theta+2\pi p)^2}(1 + (-1)^p e^{-K})$$

Mapping to height model

Fourier transform:

$$\omega_V(\theta) = \sum_{n=-\infty}^{+\infty} e^{in\theta} e^{-\frac{T}{2J}n^2}$$

Integrate out the angular dependence:

$$Z = \sum_{n_{i,j},\Delta n=0} exp(-\frac{J_*}{2}\sum_{<i,j>} n_{ij}^2 + \frac{K_*}{2}\sum_{<i,j>}(-1)^{n_{ij}})$$

Where n_{ij} satisfies the current conservation condition, and $J_* = J^{-1}$, $sinhK_* * sinhK = 1$

If we write $n_{ij} = h_i - h_j$, we have the height model:

$\mu_i = \pm 1$ depending on whether h_i is even or odd

$$Z = \sum_{h_i} exp(-\frac{J_*}{2}\sum_{<i,j>}(h_i - h_j)^2 + \frac{K_*}{2}\sum_{<i,j>}\mu_i\mu_j)$$

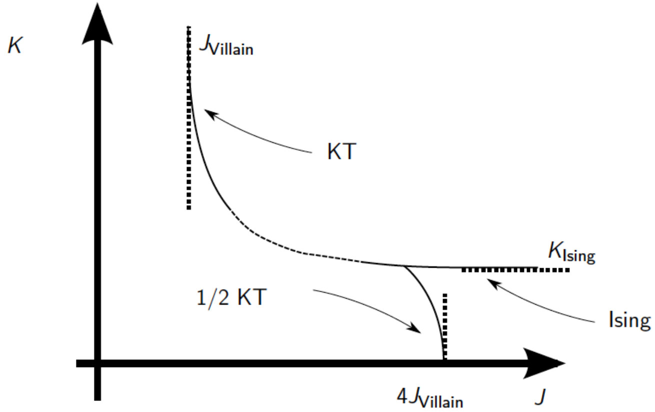
several special limits

$$Z = \sum_{h_i} exp(-\frac{J_*}{2} \sum_{<i,j>} n_{ij}^2 + \frac{K_*}{2} \sum_{<i,j>} \mu_i \mu_j)$$

- $K \to +\infty, K_* \to 0$, normal XY model
- $K = 0$, $K_* \to \infty$ the currents can only be even, replace h by $h = 2\tilde{h}$, and J by $J/4$ get the Villain model again, the critical J is 4 times bigger than Villain model
- When $J \to +\infty$, $J_* \to 0$ only the even and oddness of the current matters.
  We get an Ising model

Suggested Phase Diagram of Villain model

It has **KT, 1/2KT** and **Ising** type transitions!

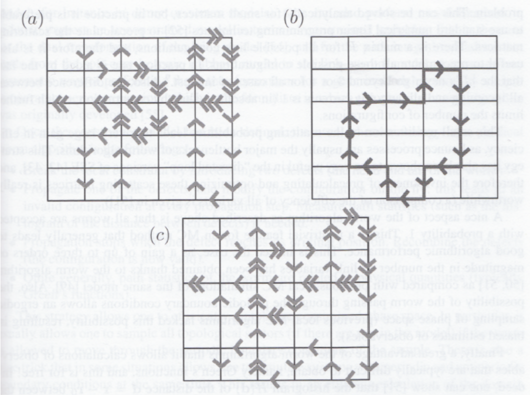


Unusual critical behavior

- Ising transition persists untill after it has met the 1/2KT transition
- Domain walls connecting half vortices effectively increasing the logrithmic interaction so they dissociate later
- This is supported by Renormalization Group study!
- Ising transition from disordered phase to quasi-long-range-oder phase

Worm algorithm

Update the system with closed current loops.
Which can be thought as the motion of a **worm**, one update finishes when the worm is closed.



Winding number and Ising sectotrs

- Winding number is how many times worms wind around one direction.
- Winding variance is proportional to the superfluid density!
- Four Ising sectors correspondes to four boundary conditions
- Crossing behavior of ratios of Ising sectors shows an Ising transition
- Worm algorithm can measure these quantities easily!

Scaling properties of winding variance and Ising sectors

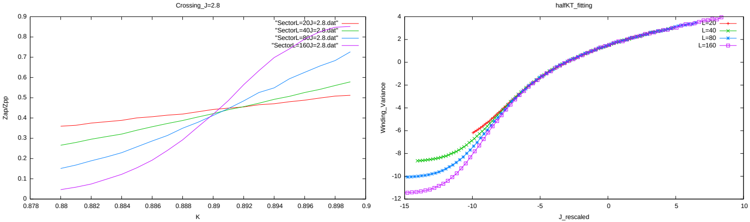
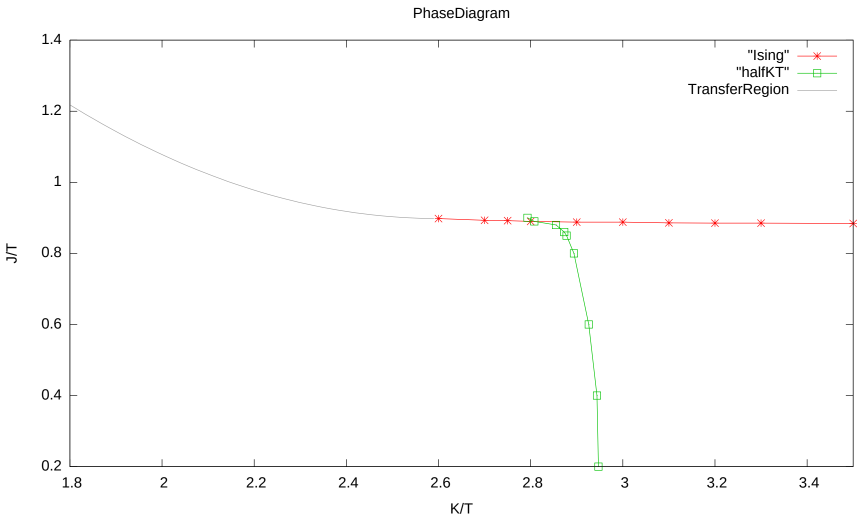


Figure: Scaling behavior for KT and Ising transition

Phase Diagram



Summary

- Three kinds of transitions in the phase diagram of vortices, half vortices and strings
- Critical fluctuations change the critical behavior
- In 3D, Vortex loops coupled to Ising gauge field
- In quantum model, atom and dimer superfluid