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Holographic superconductors at low temperatures

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OUTLINE

- Hairy black holes
- Low temperatures
- Conductivity
- Varying q
- Conclusions

◇ *G. S. and J. Therrien, JHEP* **1005** *(2010) 013 [arXiv:1003.4275]*

◇ *G. S., J. Therrien, and S. Musiri, arXiv:1011.2938*

◇ *J. Alsup, G. S., and J. Therrien, arXiv:1110.3342*

Hairy black holes

We are interested in the dynamics of a scalar field of mass m and electric charge q coupled to a $U(1)$ vector potential in the backgound of a $d+1-$ dimensional AdS black hole. The action is

$$S = \int d^{d+1}\sqrt{-g}\left[\frac{R-2\Lambda}{16\pi G} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - |(\partial_\mu - iqA_\mu)\Psi|^2 - m^2|\Psi|^2\right]$$

where $F = dA$. We shall adopt units in which $\Lambda = -\frac{d(d-1)}{2}$, $8\pi G = 1$.

To find a solution of the field equations, consider the metric *ansatz*

$$ds^2 = \frac{1}{z^2}\left[-f(z)e^{-\chi(z)}dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right]$$

where $\vec{x} \in \mathbb{R}^{d-1}$, representing an AdS black hole of planar horizon.

- AdS boundary at $z = 0$.
- choose units so that the horizon is at $z = 1$, $\therefore$ require $f(1) = 0$.
  – possible because of scaling symmetries of the system and can be done without loss of generality as long as one is careful to only consider physical quantities which are scale invariant.

Hawking temperature

$$T = -\frac{f'(1)}{4\pi} e^{-\chi(1)/2}$$

Assume scalar field is *real* function $\Psi(z)$, and potential is an electrostatic scalar potential, $A = \Phi(z)dt$.

- field equations are

$$\begin{aligned}
\Psi'' + \left[\frac{f'}{f} - \frac{\chi'}{2} - \frac{d-1}{z}\right]\Psi' + \left[\frac{q^2\Phi^2 e^{\chi}}{f^2} - \frac{m^2}{z^2 f}\right]\Psi &= 0 \\
\Phi'' + \left[\frac{\chi'}{2} - \frac{d-3}{z}\right]\Phi' - \frac{2q^2\Psi^2}{z^2 f}\Phi &= 0 \\
-\frac{d-1}{2}\chi' + z\Psi'^2 + \frac{zq^2\Phi^2\Psi^2}{f^2}e^{\chi} &= 0 \\
\frac{f}{2}\Psi'^2 + \frac{z^2}{4}\Phi'^2 e^{\chi} - \frac{d-1}{2}\frac{f'}{z} + \frac{d(d-1)}{2}\frac{f-1}{z^2} + \frac{m^2\Psi^2}{2z^2} + \frac{q^2\Psi^2\Phi^2 e^{\chi}}{2f} &= 0
\end{aligned}$$

[Hartnoll, Herzog, Horowitz]

where $z \in [0, 1]$ ($z = 1$ is horizon, $z = 0$ is boundary).

We are interested in solving the system of non-linear equations in the limit of large q (probe limit).

- expand the fields as series in $1/q$:

$$\begin{aligned}
\Psi &= \frac{1}{q}\left[\Psi_0 + \Psi_1\frac{1}{q^2} + \dots\right] \\
\Phi &= \frac{1}{q}\left[\Phi_0 + \Phi_1\frac{1}{q^2} + \dots\right] \\
f &= f_0 + f_1\frac{1}{q^2} + \dots \\
\chi &= \chi_0 + \chi_1\frac{1}{q^2} + \dots
\end{aligned}$$

- consider the zeroth order system ($q \to \infty$) first and then add first-order ($\mathcal{O}(1/q^2)$) corrections in order to obtain a physically sensible system.

Near the boundary ($z \to 0$), we have $f \to 1$, $\chi \to 0$ and so

$$\Psi \approx \Psi^{(\pm)} z^{\Delta_\pm} \quad , \quad \Phi \approx \mu - \rho z^{d-2} \quad , \quad \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$$

While a linear combination of asymptotics is allowed by the field equations, it turns out that any such combination is unstable.

Hertog and Maeda

♣ if the horizon has negative curvature, such linear combinations lead to stable configurations in certain cases.

Koutsoumbas, Papantonopoulos, Siopsis

system is labeled uniquely by the dimension $\Delta = \Delta_\pm$.

- The mass of the scalar field is bounded from below by the BF bound

$$m^2 \geq -\frac{d^2}{4}$$

- there appears to be a quantum phase transition at $m^2 = 0$.
- There is also a unitarity bound that requires $\Delta > \frac{d-2}{2}$.

μ (ρ) is the chemical potential (charge density) of the dual theory on the boundary.

leading coefficient in the expansion of the scalar yields vacuum expectation values of operators of dimension $\Delta_\pm$,

$$\langle \mathcal{O}_{\Delta_\pm} \rangle = \sqrt{2}\Psi^{(\pm)}$$

The field equations admit non-vanishing solutions for the scalar below a critical temperature T_c where these operators condense.

define

$$\Psi(z) = \frac{1}{\sqrt{2}q} b^{\Delta} z^{\Delta} F(z) \quad , \qquad b = \langle q\mathcal{O}_{\Delta}\rangle^{1/\Delta}$$

with $F(0) = 1$.

Above the critical temperature, $\Psi = 0$,

$\therefore$ field equations are solved by the AdS RN black hole with flat horizon,

$$f(z) = 1 - \left(1 + \frac{(d-2)\rho^2}{4}\right) z^d + \frac{(d-2)\rho^2}{4} z^{2(d-1)} \quad , \quad \chi(z) = 0 \quad , \quad \Phi(z) = \rho\left(1 - z^{d-2}\right)$$

Hawking temperature

$$T = \frac{d}{4\pi}\left[1 - \frac{(d-2)^2\rho^2}{4d}\right]$$

scale-invariant quantity (reduced critical temperature)

$$\hat{T} = \frac{T}{(q\rho)^{1/(d-1)}}$$

At the critical temperature, scalar field equation in RN background with $\rho = \rho_c$,

$$F'' + \left[\frac{f'}{f} + \frac{2\Delta + 1 - d}{z}\right] F' + \left[\Delta \frac{(d-\Delta)(1-f) + zf'}{z^2 f} + q^2\rho_c^2 \frac{(1 - z^{d-2})^2}{f^2}\right] F = 0$$

- For a given q, ρ_c is an eigenvalue.

To solve for large q (probe limit), expand

$$F = F_0 + F_1 \frac{1}{q^2} + \ldots \quad , \qquad \rho = \frac{1}{q}\left[\rho_0 + \rho_1 \frac{1}{q^2} + \ldots\right]$$

At zeroth order ($q \to \infty$ limit), RN background turns into an AdS Schwarzschild black hole, so

$$f_0(z) = 1 - z^d \quad , \qquad \chi_0 = 0$$

$\therefore$ scalar field equation at critical temperature

$$-F_0'' + \frac{1}{z}\left[\frac{d}{1 - z^d} - 2\Delta - 1\right] F_0' + \Delta^2 \frac{z^{d-2}}{1 - z^d} F_0 = \rho_{0c}^2 \frac{(1 - z^{d-2})^2}{(1 - z^d)^2} F_0$$

eigenvalue ρ_{0c}^2 minimizes

$$\rho_{0c}^2 = \frac{\int_0^1 dz\, z^{2\Delta-d+1}\{(1-z^d)[F_0'(z)]^2 + \Delta^2 z^{d-2}[F_0(z)]^2\}}{\int_0^1 dz\, z^{2\Delta-d+1}\frac{(1-z^{d-2})^2}{1-z^d}[F_0(z)]^2}$$

estimate by substituting the trial function

$$F_0 = F_\alpha(z) \equiv 1 - \alpha z^{d-1}$$

For $d = 3$ and $\Delta = 1, 2$, we obtain, respectively,

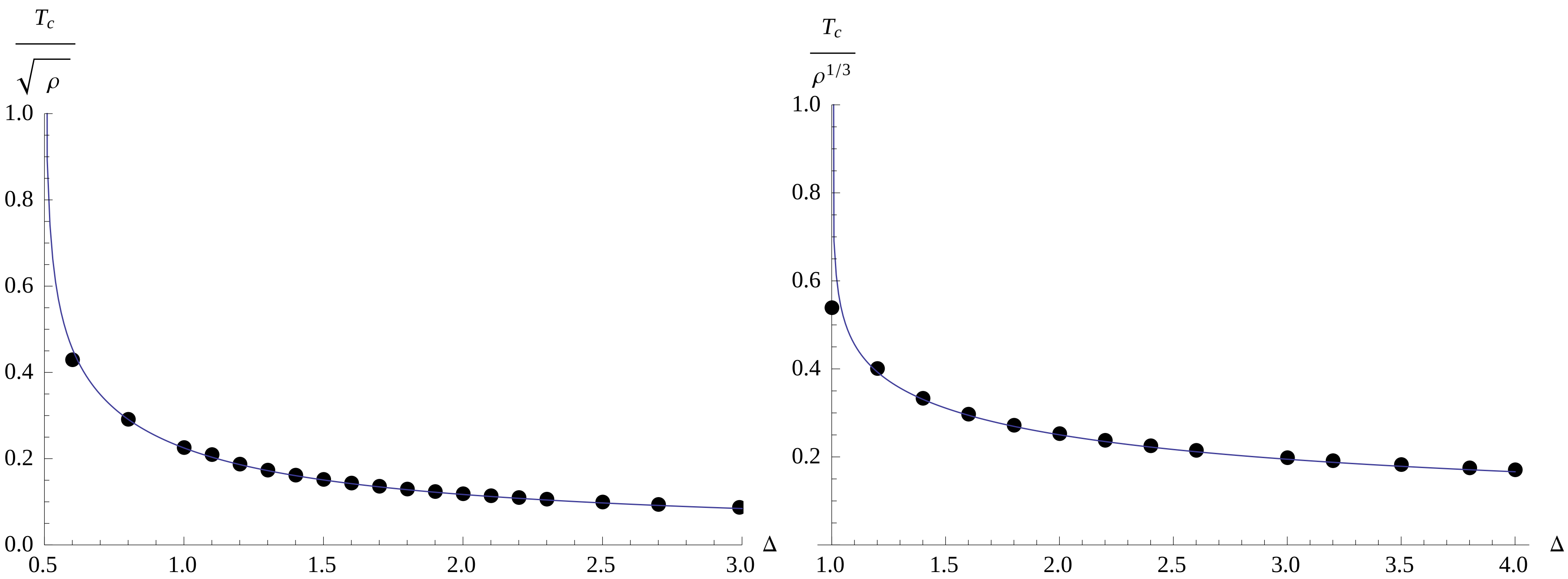
$$\rho_{0c}^2 \approx 1.27,\ 17.3 \ \ , \quad T_c \approx 0.225\sqrt{q\rho},\ 0.117\sqrt{q\rho}$$

in very good agreement with the exact $T_c = 0.226\sqrt{q\rho},\ 0.118\sqrt{q\rho}$.

For $\Delta = \frac{d}{2}$ and $d = 3, 4$, we obtain, respectively,

$$\rho_{0c}^2 \approx 6.3,\ 4.2 \ \ , \quad T_c \approx 0.15\sqrt{q\rho},\ 0.2(q\rho)^{1/3}$$

in very good agreement with the exact $T_c = 0.15\sqrt{q\rho},\ 0.25(q\rho)^{1/3}$.



The critical temperature T_c *vs* the scaling dimension Δ for $d = 3$ (left panel) and $d = 4$ (right panel). Data points represent exact values; solid line is obtained with trial function.

$\boxed{T < T_c,\ d = 3}$

Maxwell equation

$$\Phi_0'' = \frac{\langle \mathcal{O}_\Delta \rangle^2}{2} \frac{z^{2(\Delta-1)} F_0^2(z)}{1 - z^3} \Phi_0$$

Right below T_c, $\langle \mathcal{O}_\Delta \rangle$ is small. If we write

$$\Phi_0 = \rho_{0c}(1 - z) + \frac{\langle \mathcal{O}_\Delta \rangle^2}{2} \delta\Phi_0 + \dots$$

we have

$$\delta\Phi_0'' \approx \rho_{0c} \frac{z^{2(\Delta-1)} F_0^2(z)}{1 + z + z^2}$$

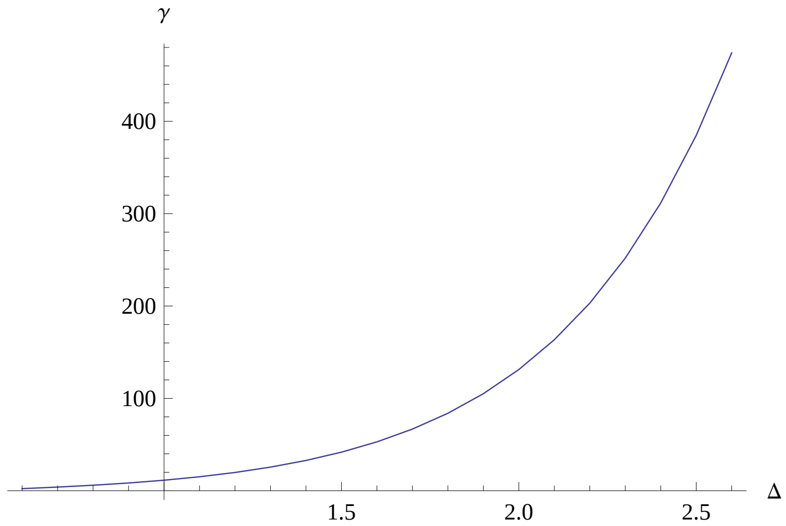
Integrating, we obtain

$$\rho = -\Phi_0'(0) \approx \rho_{0c} \left[1 + \frac{\langle \mathcal{O}_\Delta \rangle^2}{2} \mathcal{C} \right] \quad , \quad \mathcal{C} = \int_0^1 dz \frac{z^{2(\Delta-1)} F_0^2(z)}{1 + z + z^2}$$

therefore the condensate,

$$\langle \mathcal{O}_\Delta \rangle \approx \gamma T_c^\Delta \sqrt{1 - \frac{T}{T_c}} \ , \quad \gamma = \frac{2}{\sqrt{\mathcal{C}}} \left(\frac{4\pi}{3} \right)^\Delta$$

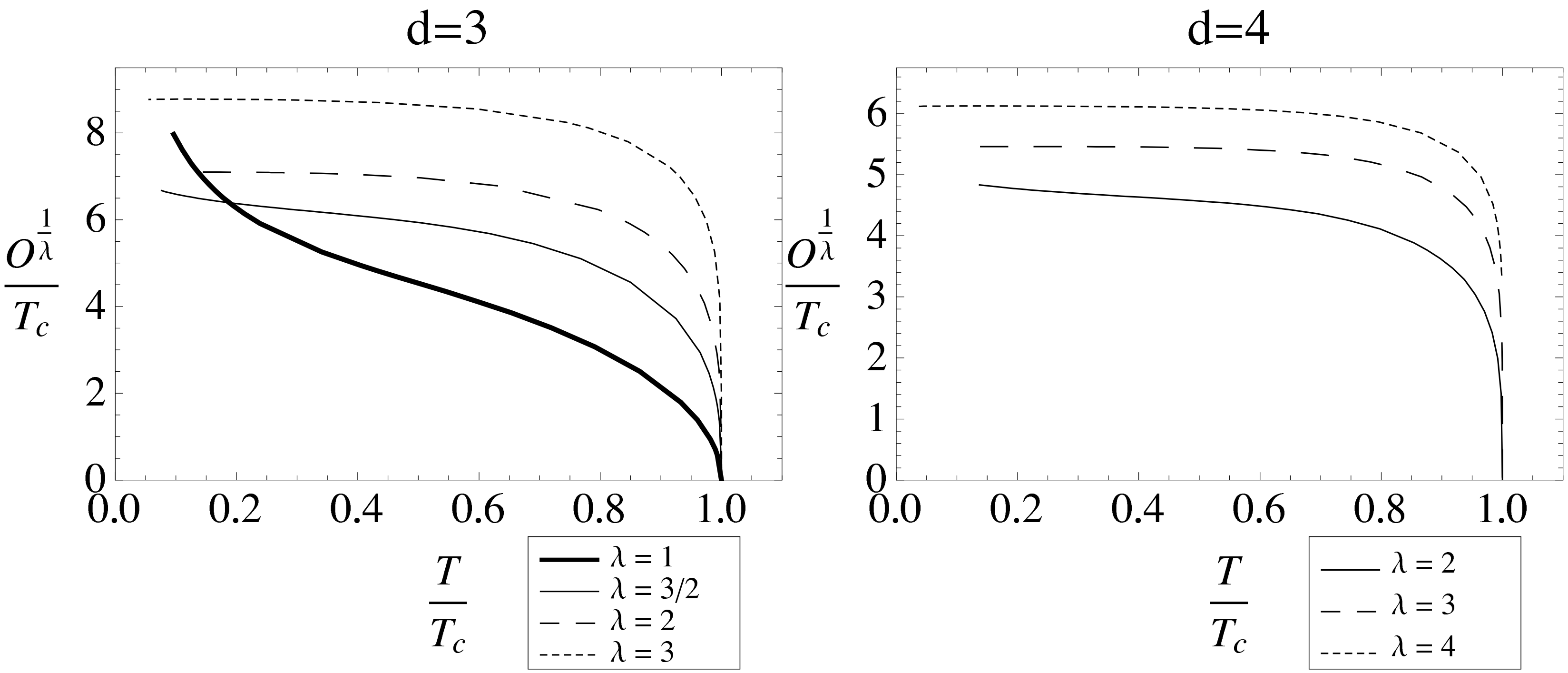
Using trial functions, we obtain for $\Delta = 1(2)$, $\gamma \approx 11.4(133)$ to be compared with exact $\gamma = 9.3(144)$.



γ *vs.* Δ in $d = 3$

$\gamma \to \infty$ as $\Delta \to 3$ (*Quantum critical point?*)

Low temperatures



Left panel: $d = 3$, right panel: $d = 4$.

[Horowitz and Roberts]

Analytic calculation

In the zero temperature limit, since radius of horizon is fixed ($z = 1$), we have $\hat{T} \to 0$ whereas T is bounded.

$\therefore\ \rho \to \infty, b \to \infty$.

We are interested in calculating scale-invariant quantities, such as

$$\frac{\langle \mathcal{O} \rangle^{1/\Delta}}{T_c} \sim \frac{b}{(q\rho)^{1/(d-1)}}$$

In the probe limit, system of zeroth-order equations,

$$\begin{aligned} -F_0'' + \frac{1}{z}\left[\frac{d}{1-z^d} - 1 - 2\Delta\right] F_0' + \frac{\Delta^2 z^{d-2}}{1-z^d} F_0 - \frac{1}{(1-z^d)^2}\Phi_0^2 F_0 &= 0 \\ \Phi_0'' - \frac{d-3}{z}\Phi_0' - \frac{b^{2\Delta} z^{2(\Delta-1)}}{1-z^d} F_0^2 \Phi_0 &= 0 \end{aligned}$$

$F_0(z)$ has a smooth limit as $T \to 0$ for $\Delta = \Delta_- \le \frac{d}{2}$,

$$F_0(z) = \frac{\Gamma^2(1-\frac{\Delta}{d})}{\Gamma(1-\frac{2\Delta}{d})} F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; 1; 1-z^d\right)$$

$\diamondsuit$ not the case for $\Delta = \Delta_+ > \frac{d}{2}$.

solve by iteration at low temperatures ($b \to \infty$ limit).

$$-F_0^{(n+1)\prime\prime} + \frac{1}{z}\left[\frac{d}{1-z^d} - 1 - 2\Delta\right] F_0^{(n+1)\prime} + \frac{\Delta^2 z^{d-2}}{1-z^d} F_0^{(n+1)} = \frac{\mu^2}{(1-z^d)^2}[\hat{\Phi}_0^{(n+1)}]^2 F_0^{(n)}$$

$$\hat{\Phi}_0^{(n+1)\prime\prime} - \frac{d-3}{z}\hat{\Phi}_0^{(n+1)\prime} - \frac{b^{2\Delta} z^{2(\Delta-1)}}{1-z^d}[F_0^{(n)}]^2 \hat{\Phi}_0^{(n+1)} = 0$$

starting with

$$F_0^{(0)}(z) = 1 \ , \quad \hat{\Phi}_0^{(0)}(z) = 0$$

We defined

$$\Phi_0(z) = \mu \hat{\Phi}_0(z) \ , \quad \hat{\Phi}_0(0) = 1$$

where μ is the chemical potential.
At the nth step, we obtain for the scalar field

$$F_0^{(n+1)}(z) = \mathcal{F}_1(z)\left[1 + \mu^2 \int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_2(z')[\hat{\Phi}_0^{(n+1)}(z')]^2 F_0^{(n)}(z')\right]$$
$$-\mathcal{F}_2(z)\mu^2 \int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_1(z')[\hat{\Phi}_0^{(n+1)}(z')]^2 F_0^{(n)}(z')$$

where

$$\mathcal{F}_1(z) = F\left(\frac{\Delta}{d}, \frac{\Delta}{d}; \frac{2\Delta}{d}; z^d\right) \ , \quad \mathcal{F}_2(z) = \frac{z^{d-2\Delta}}{d-2\Delta} F\left(1-\frac{\Delta}{d}, 1-\frac{\Delta}{d}; 2-\frac{2\Delta}{d}; z^d\right)$$

BC: $F_0^{(n+1)}(0) = 1$. At the horizon, this function diverges. Demanding regularity at $z=1$ fixes μ.

For $n=0$, we obtain for the electrostatic potential

$$\hat{\Phi}_0^{(1)}(z) = \frac{2}{\Gamma(\nu)(2\Delta)^\nu}(bz)^{\frac{d-2}{2}}\left[K_\nu\left(\frac{(bz)^\Delta}{\Delta}\right) - \frac{K_\nu\left(\frac{b^\Delta}{\Delta}\right)}{I_\nu\left(\frac{b^\Delta}{\Delta}\right)}I_\nu\left(\frac{(bz)^\Delta}{\Delta}\right)\right] \quad , \quad \nu = \frac{d-2}{2\Delta}$$

second Bessel function has an exponentially small coefficient, $\mathcal{O}(\sim e^{-2b^\Delta/\Delta})$, and can be neglected at low T.

Charge density

$$\frac{\rho_0}{b^{d-2}} = -\frac{\mu}{(2\Delta)^{2\nu}}$$

For the scalar field we obtain

$$F_0^{(1)}(z) = \mathcal{F}_1(z)\left[1+\mu^2\int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_2(z')[\hat{\Phi}_0^{(1)}(z')]^2\right] - \mathcal{F}_2(z)\mu^2\int_0^z \frac{dz'}{1-(z')^d}(z')^{2\Delta+1-d}\mathcal{F}_1(z')[\hat{\Phi}_0^{(1)}(z')]^2$$

logarithmic singularity at the horizon

$$F_0^{(1)}(z) \approx -\left[\frac{\Gamma(\frac{2\Delta}{d})}{\Gamma^2(\frac{\Delta}{d})}(1+\mu^2 a_2) - \frac{\Gamma(2-\frac{2\Delta}{d})}{(d-2\Delta)\Gamma^2(1-\frac{\Delta}{d})}\mu^2 a_1\right]\ln(1-z)$$

Demanding regularity at the horizon, fixes the chemical potential,

$$\frac{1}{\mu^2} = \frac{\Gamma(2-\frac{2\Delta}{d})\Gamma^2(\frac{\Delta}{d})}{(d-2\Delta)\Gamma(\frac{2\Delta}{d})\Gamma^2(1-\frac{\Delta}{d})} a_1 - a_2$$

Explicitly,

$$a_1 = \frac{1}{b^{2\Delta+2-d}} \frac{(d-2)\Gamma(1-\nu)}{(2\Delta)^{2\nu}\Gamma(\nu)} + \dots \ , \quad a_2 = \frac{1}{b^2} \frac{\sqrt{\pi}\Delta^{\frac{2}{\Delta}-1}\Gamma(\frac{1}{\Delta})\Gamma(\frac{d-1}{\Delta})\Gamma(\frac{d}{2\Delta})}{(d-2\Delta)\Gamma^2(\nu)2^{2\nu}\Gamma(\frac{d+\Delta}{2\Delta})} + \dots$$

Evidently, for $\Delta < \frac{d}{2}$, $a_2/a_1 \to 0$ as $b \to \infty$, therefore

$$\mu^2 \approx \mathcal{C} b^{2\Delta+2-d} \ , \quad \mathcal{C} = \frac{(d-2\Delta)(2\Delta)^{2\nu}\Gamma(\nu)\Gamma(\frac{2\Delta}{d})\Gamma^2(1-\frac{\Delta}{d})}{(d-2)\Gamma(1-\nu)\Gamma(2-\frac{2\Delta}{d})\Gamma^2(\frac{\Delta}{d})}$$

It is easily seen (using standard hypergeometric identities) that the low temperature expression reduces to the $T = 0$ one as $b \to \infty$.

charge density

$$\rho_0 \sim b^{\frac{d}{2}+\Delta-1}$$

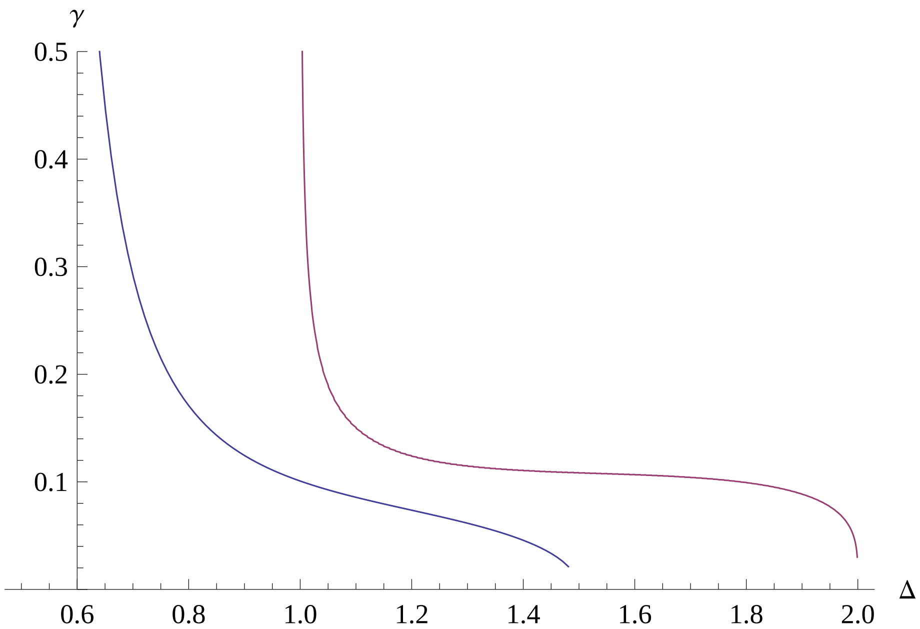
Using

$$\frac{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} \sim b\rho_0^{-\frac{1}{d-1}} \quad , \quad \frac{T}{T_c} \sim \rho_0^{-\frac{1}{d-1}}$$

we finally obtain

$$\frac{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} = \gamma \left(\frac{T}{T_c} \right)^{-\frac{d/2-\Delta}{d/2+\Delta-1}}$$

- condensate diverges as $T \to 0$.
- exponent depends on the dimensions of the operator and spacetime.
- constant of proportionality γ can be found analytically.



The parameter γ *vs* Δ. Curve on left (right) is for $d = 3$ ($d = 4$).

As $\Delta \to \frac{d}{2}$ (BF bound), $\gamma \to 0$,

$\therefore$ power law behavior changes

Letting $\Delta = \frac{d}{2} - \varepsilon$,

$$\frac{1}{\mu^2} = \frac{(d-2)\Gamma(\frac{2}{d})}{d^{2(1-\frac{2}{d})}\Gamma(1-\frac{2}{d})} \left[\frac{1}{2\varepsilon b^{2-2\varepsilon}} - \frac{1}{2\varepsilon b^2} + \dots \right]$$

and taking the limit $\varepsilon \to 0$, we deduce at the BF bound

$$\frac{\mu^2}{b^2} = \frac{d^{2(1-\frac{2}{d})}\Gamma(1-\frac{2}{d})}{(d-2)\Gamma(\frac{2}{d})[\ln b + \beta_d + o(b^0)]}$$

where β_d is a constant that depends on the dimension and is easily computed (e.g., for $d = 3$, $\beta_3 \approx 1.75$).

For the charge density,

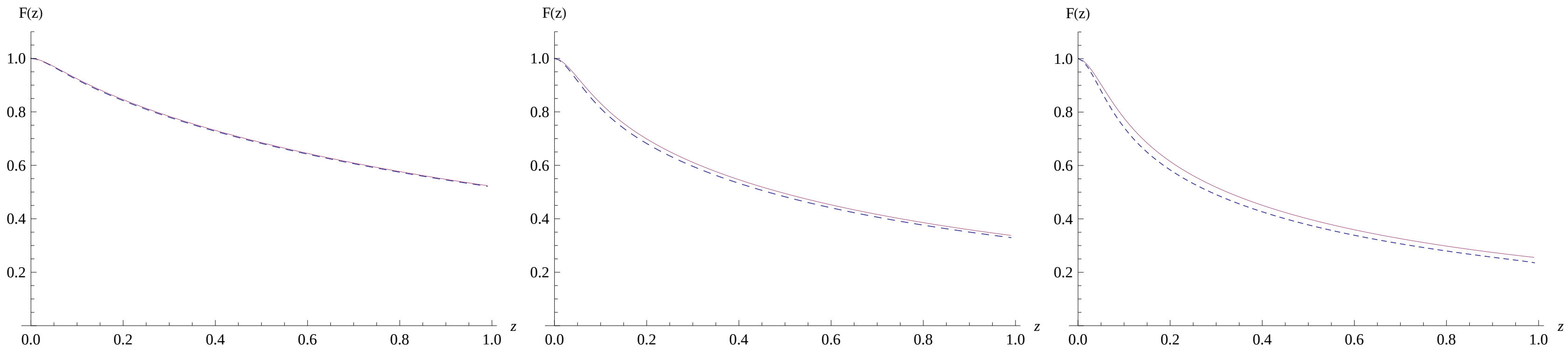
$$\rho_0 \sim b^{d-1}(\ln b)^{-1/2}$$

and the condensate

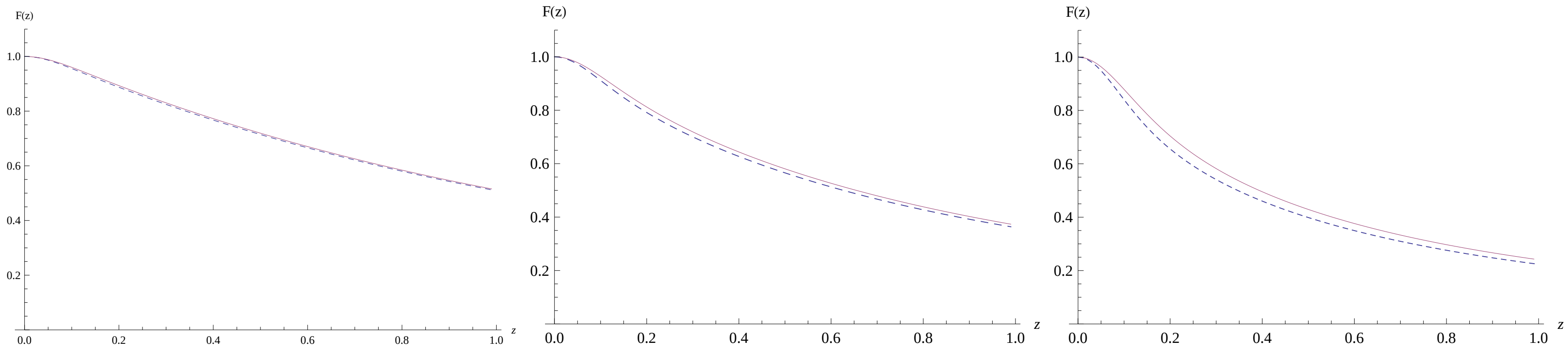
$$\frac{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}{T_c} \sim (\ln b)^{\frac{1}{2(d-1)}} \sim \left(\ln \frac{T_c}{T}\right)^{\frac{1}{2(d-1)}}$$

showing that the condensate diverges at the BF bound, albeit very mildly.

- mild divergence was missed in earlier numerical studies.

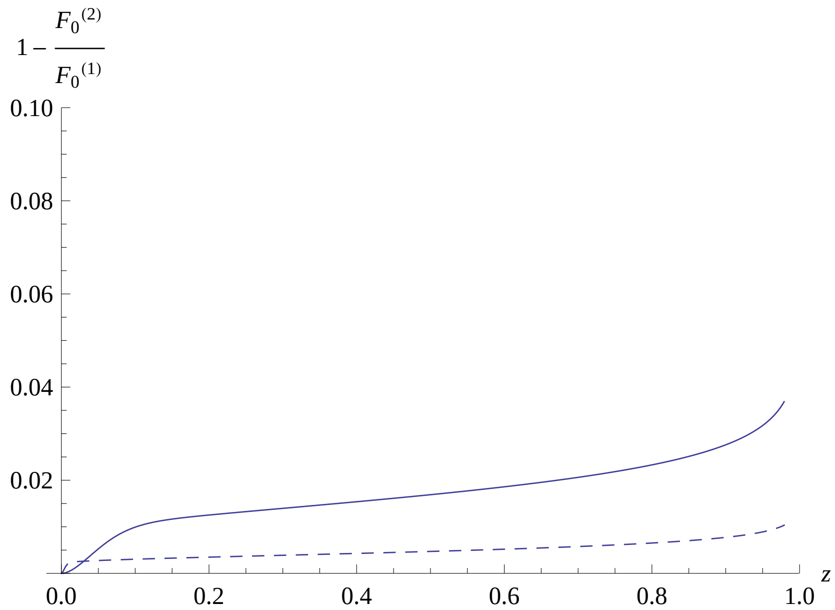


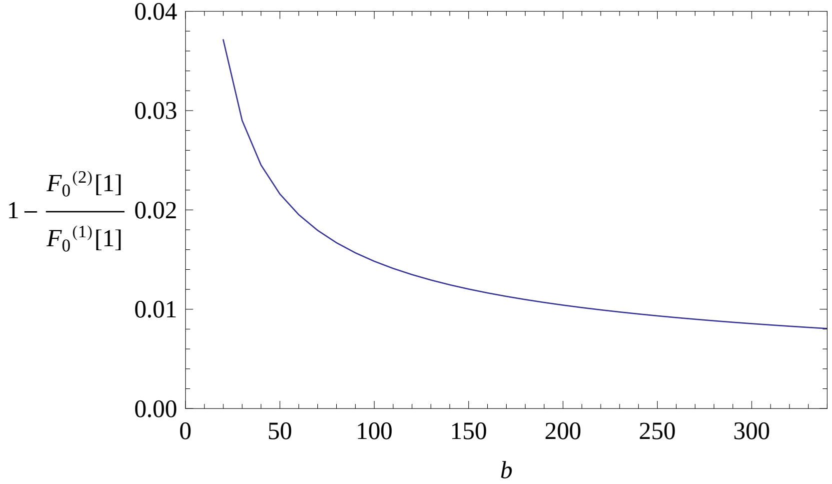
The field F for $\Delta = 1.2$ (left panel), 1.4 (middle panel), 1.5 (right panel) and $d = 3$. First-order analytic results (dashed lines) compared with exact numerical results (almost indistinguishable solid lines) at $T/T_c \approx 0.1$.



The field F for $\Delta = 1.6$ (left panel), 1.8 (middle panel), 2 (right panel) and $d = 4$.

Corrections





Corrections are $\lesssim$ 1% for $T/T_c \lesssim 0.1$ (left panel) and vanish (subleading) as $T \to 0$ (right panel).

Above the BF bound ($\Delta > d/2$), as $T \to 0$, we have $F_0 \approx 1$ near the boundary ($z \lesssim 1/b$), but asymptotically (large $z \gtrsim 1/b$), $F_0 \sim (bz)^{d-2\Delta}$, which does not have a smooth limit as $T \to 0$.

- cannot apply perturbation theory
- instead, one can approximate F_0 by

$$F_0(z) = \begin{cases} 1 & , \quad z \le \alpha/b \\ \left(\frac{bz}{\alpha}\right)^{d-2\Delta} & , \quad z > \alpha/b \end{cases}$$

  and find α by a variational method.

first-order corections in a $1/q^2$ expansion.

For $\Delta < d/2$, it is necessary to include corrections to obtain a physical system at low temperatures,

$\because$ in the $q \to \infty$ limit the condensate diverges as $T \to 0$.

At first order, we obtain for the functions determining the metric,

$$\begin{aligned} z f_1' - d f_1 &= \frac{(bz)^{2\Delta}}{4(d-1)} \left[\left(m^2 + \Delta^2 f_0 + \frac{z^2 \Phi_0^2}{(bz)^{2\Delta}} \right) F_0^2 + 2\Delta z f_0 F_0 F_0' + z^2 f_0 {F_0'}^2 + \frac{z^4}{(bz)^{2\Delta}} {\Phi_0'}^2 \right] \\ z\chi_1' &= \frac{(bz)^{2\Delta}}{d-1} \left[\left(\Delta^2 + \frac{z^2 \Phi_0^2}{f_0^2} \right) F_0^2 + 2\Delta z F_0 F_0' + z^2 {F_0'}^2 \right] \end{aligned}$$

Solution

$$f_1(z) = -\frac{\Delta}{4(d-1)} (bz)^{2\Delta} \left[2 - z^d - z^{d-2\Delta} \right] + \dots \,, \quad \chi_1(z) = -\frac{\Delta}{2(d-1)} (bz)^{2\Delta} + \dots$$

For the temperature, we deduce the first-order expression

$$T = \frac{d}{4\pi} \left[1 + \frac{\Delta^2}{2d(d-1)} \frac{b^{2\Delta}}{q^2} + \dots \right]$$

temperature receives a positive correction away from the probe limit.

probe limit fails: for the expansion in $1/q^2$ to be valid, we ought to have

$$b \lesssim q^{1/\Delta}$$

For a given q, this places a lower bound on the temperature.

- While $T = 0$ is unattainable for finite q, T can be made arbitrarily low by choosing a sufficiently large q.
- even though the probe limit ($q \to \infty$) is not a physical system, its properties are a good approximation to corresponding properties of physical systems (of finite q).
  - approximation becomes better with increasing q and the $1/q^2$ expansion is valid.

Conductivity

Low Temperature Conductivity

We shall obtain the conductivity σ as a function of the rescaled frequency

$$\hat{\omega} = \frac{\omega}{b} = \frac{\omega}{\langle \mathcal{O}_\Delta \rangle^{1/\Delta}}$$

The function $\sigma(\hat{\omega})$ has a well-defined limit as $q \to \infty$ (probe limit) down to zero temperature even though the condensate $\langle \mathcal{O}_\Delta \rangle$ diverges.

- The probe limit, which is not a physical state at low temperatures, can be arbitrarily well approximated by physical states of sufficiently large q. The conductivity of these states can be obtained as a $1/q^2$ expansion with the conductivity in the probe limit serving as the zeroth order term in the expansion.

$\boxed{d = 3}$

The conductivity on the AdS boundary is found by applying a sinusoidal electromagnetic perturbation in the bulk of frequency ω obeying the wave equation

$$-\frac{d^2 A}{dr_*^2} + VA = \omega^2 A \ , \quad V = \frac{2q^2}{z^2} f \Psi^2$$

where A is any component of the perturbing electromagnetic potential along the boundary.

- ingoing boundary condition at the horizon

$$A \sim e^{-i\omega r_*} \sim (1-z)^{-i\omega/3}$$

as $z \to 1$ ($r_* \to -\infty$), where r_* is the tortoise coordinate

$$r_* = \int \frac{dz}{f(z)} = \frac{1}{6}\left[\ln\frac{(1-z)^3}{1-z^3} - 2\sqrt{3}\tan^{-1}\frac{\sqrt{3}z}{2+z}\right]$$

with the integration constant chosen so that the boundary is at $r_* = 0$.
wave equation reads

$$\frac{d}{dz}\left[(1-z^3)\frac{dA}{dz}\right] - \left[b^{2\Delta}z^{2\Delta-2}F^2(z) - \frac{\omega^2}{1-z^3}\right]A = 0$$

set

$$A = (1-z)^{-i\omega/3}e^{-i\omega z/3}\mathcal{A}(z)$$

- included a factor $e^{-i\omega z/3}$ so that only $\mathcal{A}(z)$ contributes to conductivity.

wave equation becomes

$$-3(1-z^3)\mathcal{A}'' + z\left[9z - 2(1+z+z^2)i\omega\right]\mathcal{A}'$$
$$+\left[3b^{2\Delta}z^{2\Delta-2}F^2(z) - (1+2z+3z^2)i\omega - \frac{(3+2z+z^2)(3+z+z^2+z^3)}{3(1+z+z^2)}\omega^2\right]\mathcal{A} = 0$$

Regularity of $\mathcal{A}$ at horizon ($z=1$) implies the b.c.

$$(3-2i\omega)\,\mathcal{A}'(1) + \left(b^{2\Delta}F^2(1) - 2i\omega - \frac{4\omega^2}{3}\right)\mathcal{A}(1) = 0$$

In zero temperature limit, $b\to\infty$, rescale $z\to z/b$, and solve as a series expansion in $1/q^2$.

zeroth order term (probe limit) by replacing F by F_0, For $\Delta \le \frac{3}{2}$,

$$-\mathcal{A}'' + \left[z^{2\Delta-2} - \widehat{\omega}^2\right]\mathcal{A} = 0$$

where we used $F(z/b)\to F(0)=1$, as $b\to\infty$. For $1<\Delta\le\frac{3}{2}$, there are two linearly independent solutions, $\mathcal{A}_\pm$, distinguished by their asymptotic behavior,

$$\mathcal{A}_\pm \sim e^{\pm\frac{1}{\Delta}z^\Delta}\ , \quad z\to\infty$$

The general solution can be written as a linear combination,

$$\mathcal{A} = c^{+}\mathcal{A}_{+} + c^{-}\mathcal{A}_{-}$$

Applying the b.c. at horizon, we deduce

$$\frac{c^{+}}{c^{-}} \sim e^{-\frac{2}{\Delta}b^{\Delta}}$$

so at zero temperature

$$c^{+} = 0$$

i.e., $\mathcal{A} \to 0$ as $z \to \infty$.

$\boxed{\Delta = 1}$

Exact solution

$$\mathcal{A}(z) = e^{-\sqrt{1-\widehat{\omega}^{2}}bz}$$

whereas $\mathcal{A}_{+}(z) = e^{+\sqrt{1-\widehat{\omega}^{2}}bz}$, with arbitrary normalization, where we restored the scaling parameter b.

$T = 0$ conductivity in probe limit

$$\sigma(\widehat{\omega}) = \frac{i}{\widehat{\omega}}\sqrt{1-\widehat{\omega}^2}$$

$\Re\sigma = 0$ for $\widehat{\omega} \le 1$, $\therefore$ gap

$$\omega_g = \langle \mathcal{O}_1 \rangle$$

At low T, corrections $\sim e^{-2b}$, therefore

$$\Re\sigma \sim e^{-E_g/T} \quad , \qquad E_g = \frac{3\langle \mathcal{O}_1 \rangle}{2\pi} \approx 0.48\omega_g$$

to be compared with the BCS result for energy gap $E_g = \omega_g/2 = 0.50\omega_g$.

$$\boxed{\Delta = \frac{3}{2}}$$

$$\mathcal{A}(z) = \mathcal{A}_-(z) = \mathrm{Ai}(bz - \widehat{\omega}^2)$$

whereas $\mathcal{A}_+(z) = \mathrm{Bi}(bz - \widehat{\omega}^2)$.

At $T = 0$ in the probe limit, the quasinormal frequencies have moved to the real axis yielding an infinite set of normal frequencies which are solutions of

$$\mathrm{Ai}(-\hat{\omega}^2) = 0$$

- infinite tower of real frequencies given by the zeroes of the Airy function.

The $T = 0$ conductivity in the probe limit is

$$\sigma(\hat{\omega}) = \frac{i}{\hat{\omega}} \frac{\mathrm{Ai}'(-\hat{\omega}^2)}{\mathrm{Ai}(-\hat{\omega}^2)}$$

normal modes are the poles of conductivity.

- At zero temperature, $\Re\sigma = 0$, except at the poles of $\Im\sigma$ where $\Re\sigma$ diverges as a δ-function. Gap

$$\omega_g \approx 1.5 \langle \mathcal{O}_{3/2} \rangle^{2/3}$$

At low temperatures,

$$\Re\sigma \sim e^{-\frac{4}{3}b^{3/2}} = e^{-(E_g/T)^{3/2}} \ , \quad E_g = \frac{1}{\pi}\left(\frac{3}{4}\right)^{1/3} \langle \mathcal{O}_{3/2} \rangle^{2/3} \approx 0.2\omega_g$$

to be compared with the BCS result $E_g = 0.5\omega_g$.

For first-order correction $\delta\mathcal{A}$,

$$-\delta\mathcal{A}'' + [z - \hat{\omega}^2]\delta\mathcal{A} = -\frac{1}{3(1-z^3)}\mathcal{H}_1\mathcal{A}$$

where

$$\begin{aligned}\mathcal{H}_1 &= z\left[9z - 2(1+z+z^2)i\omega\right]\frac{d}{dz} + 3b^3z(2F_1(z)+z^3) - (1+2z+3z^2)i\omega \\ &\quad + \frac{z^2(1-15z-12z^2-10z^3)}{3(1+z+z^2)}\omega^2\end{aligned}$$

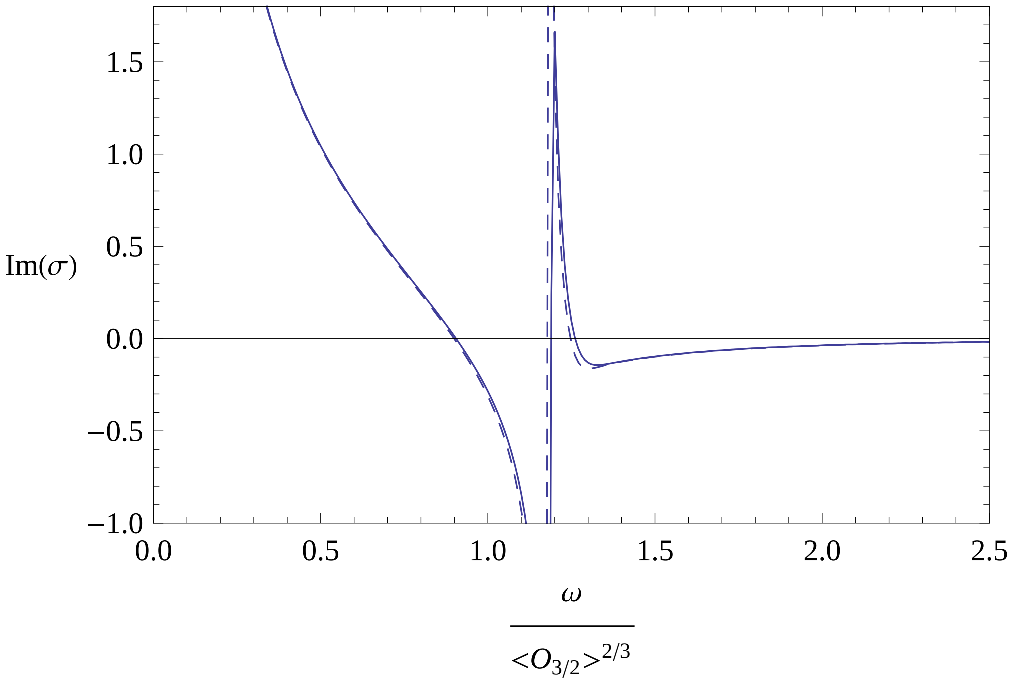
$T = 0$ real frequencies get shifted at low $T > 0$ away from the real axis,

$$\delta\hat{\omega} = \frac{\pi \mathrm{Bi}(-\hat{\omega}^2)}{3\hat{\omega}\mathrm{Ai}'(-\hat{\omega}^2)}\int_0^1 \frac{dz}{1-z^3}\mathrm{Ai}(bz-\hat{\omega}^2)\mathcal{H}_1\mathrm{Ai}(bz-\hat{\omega}^2)$$

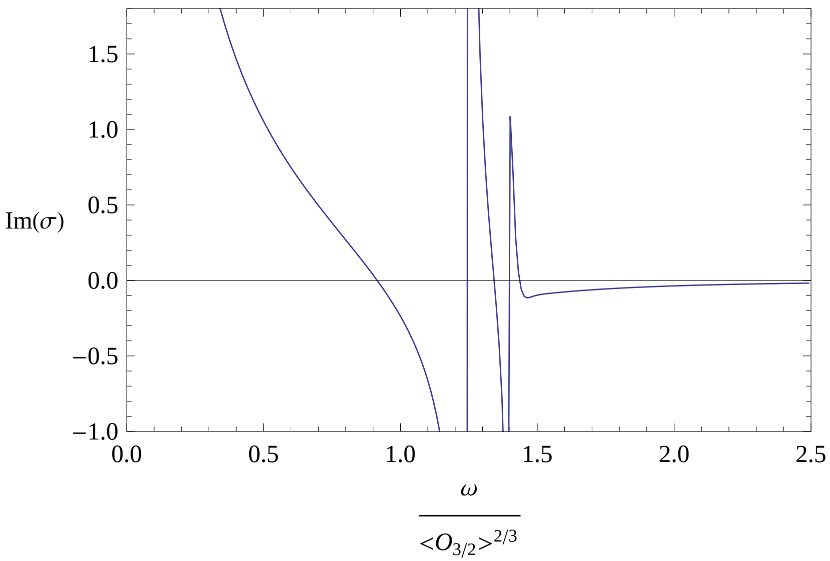
valid for low frequencies.

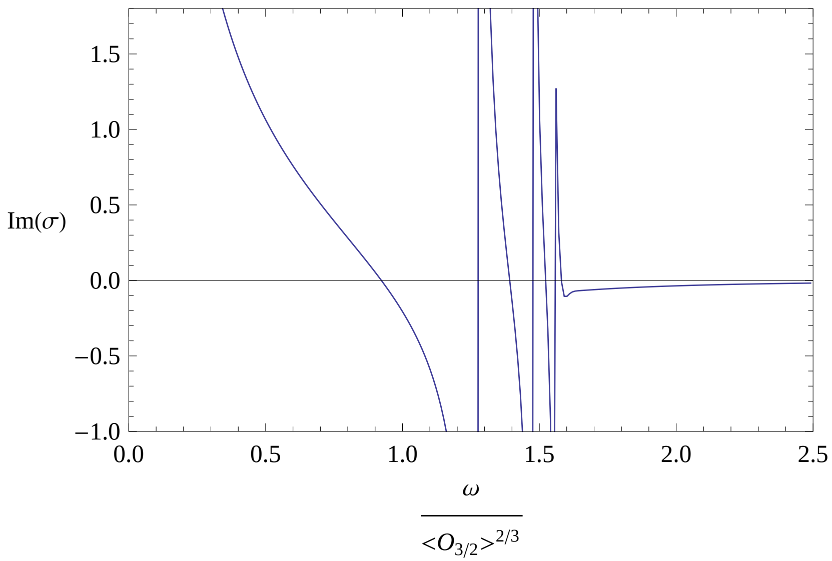
- As we heat up the system, most modes disappear and we are left with a finite number of quasinormal modes. Their number decreases as we increase the temperature.
- Conversely, as we cool down the system, modes shift to the real axis ($\delta\hat{\omega} \to 0$ as $T \to 0$), and at $T = 0$ we obtain an infinite number of real frequencies.

As the mode frequency approaches the real axis, the corresponding spike in the plot of the imaginary part of the conductivity becomes more pronounced.

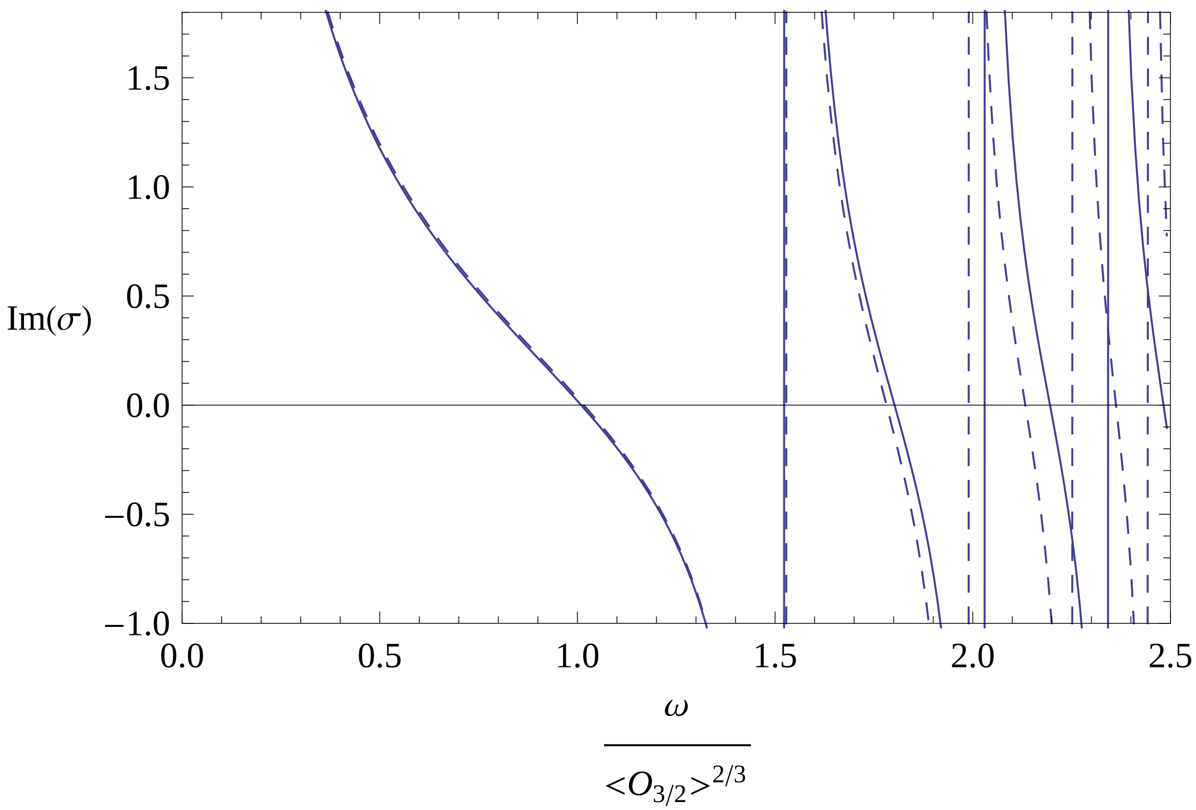


The imaginary part of the conductivity in $d = 3$ using the analytic expression for the scalar field (dotted line) compared with the exact numerical solution (solid line) at $\frac{T}{T_c} \approx .1$





The imaginary part of the conductivity *vs.* frequency in $d = 3$ using the analytic expression for F at $\frac{T}{T_c} \approx .05$ (left), .04 (right). As the temperature decreases, poles move on to the real axis.



Comparison of the imaginary part of the conductivity in $d = 3$ using the analytic expression for F at $\frac{T}{T_c} \approx .01$ (dotted line) and the zero temperature limit (solid line).

For $\Delta > 3/2$, the potential is

$$V = b^{2\Delta} z^{2\Delta - 2}(1 - z^3)F(bz)$$

It attains a maximum of order $b^{2(2-\Delta)}$ for $\Delta < 2$. Therefore, at zero temperature it has infinite height. However, the width becomes infinitely narrow leading to a finite tower of poles for the conductivity (quasinormal modes). In the zero temperature limit, the number of modes increases as one approaches the BF bound and decreases away from it.

$\boxed{d = 4}$

At BF bound, $\Delta = 2$, in the probe limit, wave equation

$$A'' - \frac{1}{z}A' - [b^4 z^2 - \omega^2]A = 0$$

whose acceptable solution can be written in terms of a Whittaker function,

$$A = W_{\frac{\widehat{\omega}^2}{4}, \frac{1}{2}}(b^2 z^2)$$

(The other solution diverges as $z \to \infty$.)

At the boundary ($z \to 0$), it has a logarithmic divergence which we need to subtract before we can calculate QNMs and the conductivity

[Horowitz and Roberts]

The conductivity is then given by

$$\sigma(\widehat{\omega}) = \frac{2}{i\widehat{\omega}} \frac{A_2}{A_0} + \frac{i\widehat{\omega}}{2}$$

with an arbitrarily chosen cutoff, where

$$A(z) = A_0 + A_2 b^2 z^2 - A_0 \frac{\widehat{\omega}^2}{2} b^2 z^2 \ln(b^2 z^2) + \dots$$

Using the expansion for small arguments,

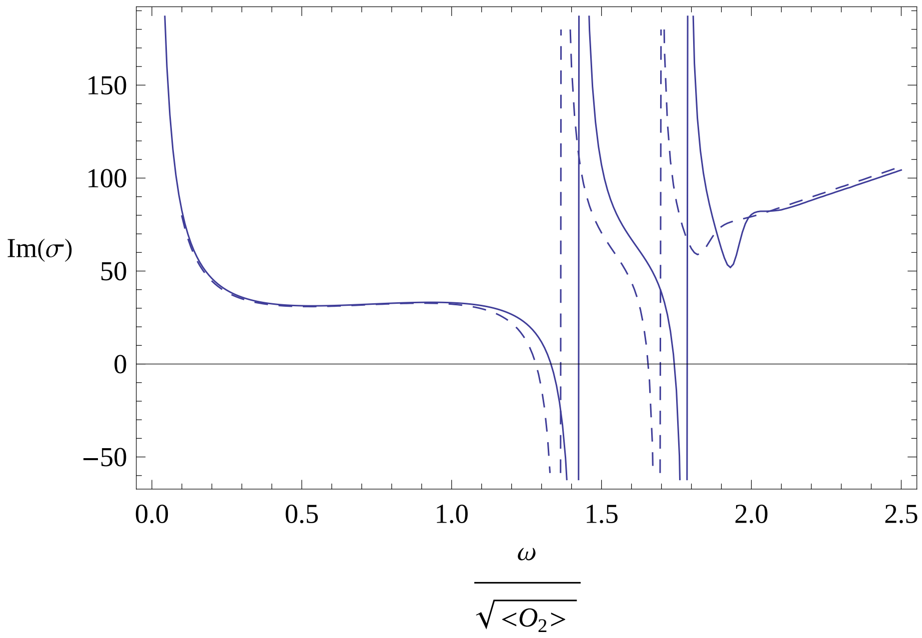
$$W_{\frac{\widehat{\omega}^2}{4},\frac{1}{2}}(b^2z^2) = -\frac{2}{\widehat{\omega}^2\Gamma(-\widehat{\omega}^2/4)}\left\{1 - \left[1 + \widehat{\omega}^2\left(2\gamma - 1 + \ln(b^2z^2) + \psi(1-\widehat{\omega}^2/4)\right)\right]\frac{b^2z^2}{2}\right\} + \dots$$

we deduce the $T = 0$ conductivity in the probe limit

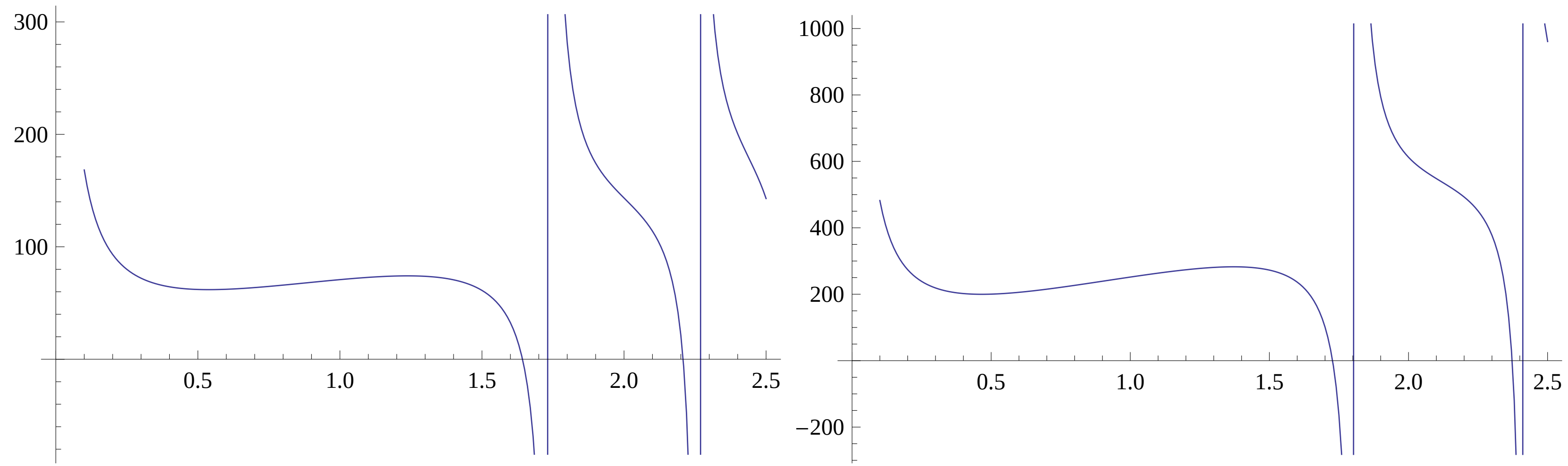
$$\sigma(\widehat{\omega}) = \frac{1}{i\widehat{\omega}} + i\widehat{\omega}\left[2\gamma - \frac{1}{2} + \psi(1-\widehat{\omega}^2/4)\right]$$

We have a pole at $\omega = 0$, as expected and an infinite tower of real poles determined by the poles of the digamma function,

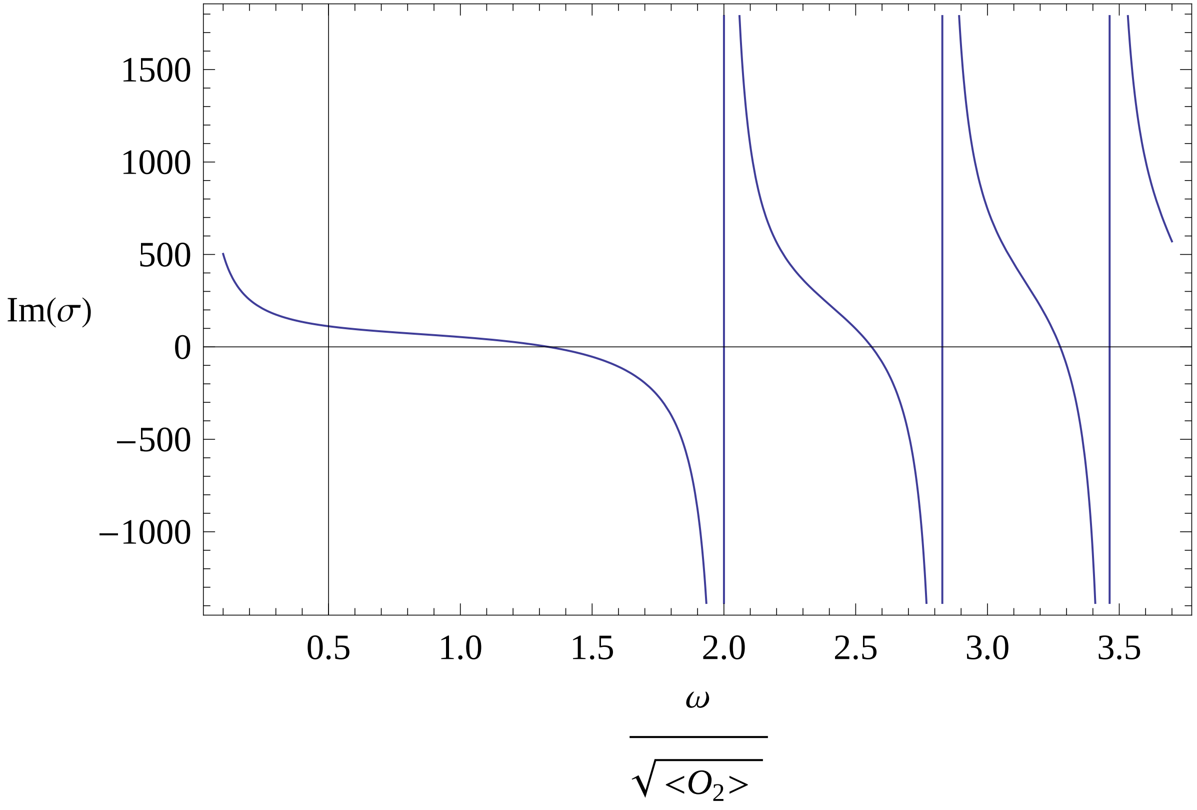
$$\widehat{\omega} = \frac{\omega_n}{\langle\mathcal{O}\rangle^{1/2}} = 2\sqrt{n} \ , \quad n = 0, 1, 2, \dots$$



$\Im\sigma$ in $d = 4$ using the analytic expression for the scalar field (dotted line) compared with the exact numerical solution (solid line) at $\frac{T}{T_c} \approx .17$



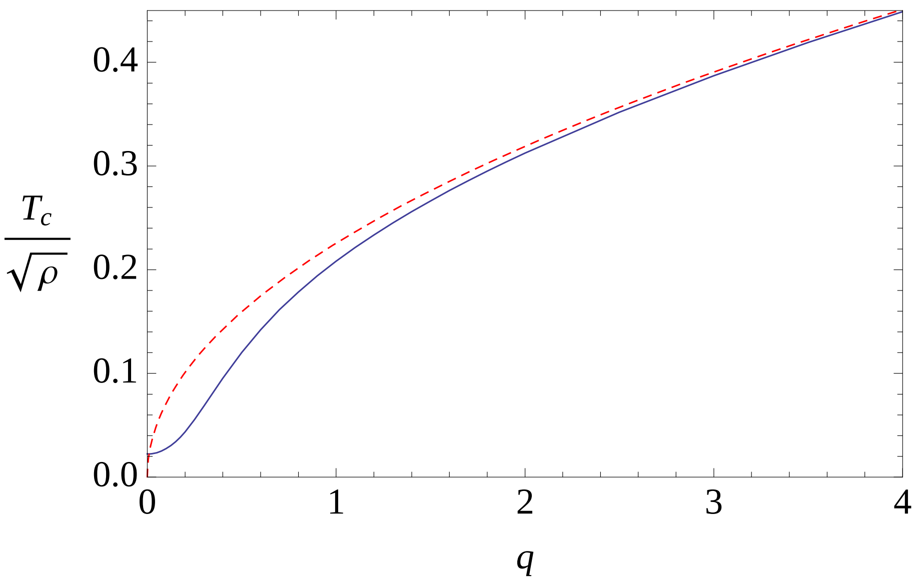
$\Im\sigma$ in $d = 4$ using the analytic expression for F at $\frac{T}{T_c} \approx .1$ (left), .04 (right).

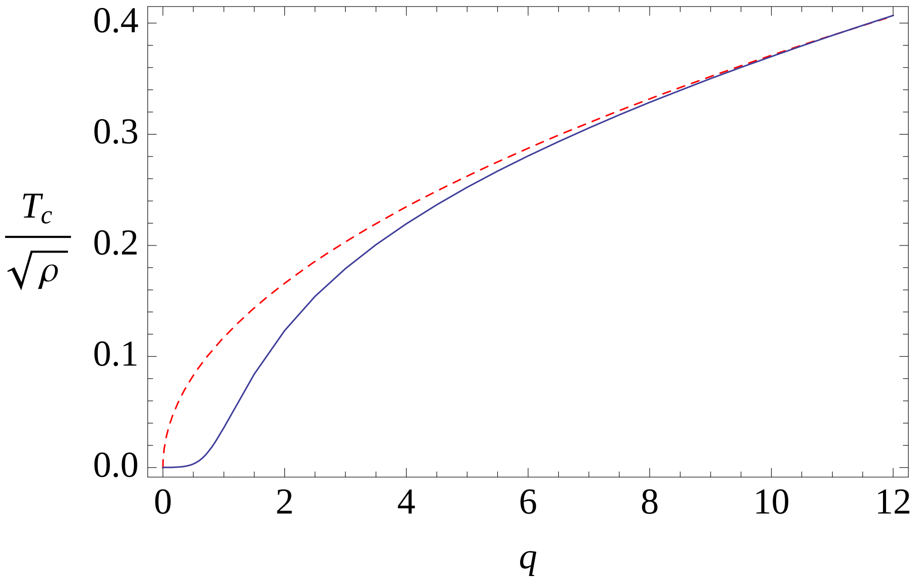


$\Im\sigma$ at $T = 0$ in $d = 4$.

Varying q

Quantum Critical Point (Varying q)





$\Delta = 1$ (left panel), $\Delta = 2$ (right panel). *[Hartnoll, Herzog, Horowitz]*

GOAL

- phase transition between $q = 0$ and $q \to \infty$?
  - quantum critical point at $q \sim 1$?
- calculate analytically (numerical analysis fails).

Above the critical temperature ($T \geq T_c$), we have $\Psi = 0$ and $\chi = 0$. The field equations reduce to

$$\begin{aligned} \Psi'' + \left[\frac{f'}{f} - \frac{2}{z}\right]\Psi' + \left[\frac{q^2\Phi^2}{f^2} - \frac{m^2}{z^2 f}\right]\Psi &= 0 \\ \Phi'' &= 0 \\ \frac{z^2}{4}\Phi'^2 - \frac{f'}{z} + \frac{3(f-1)}{z^2} &= 0 \end{aligned}$$

The first equation has a non-zero solution only at $T = T_c$.

The other two equations yield the RN black hole

$$f(z) = 1 - \left(1 + \frac{\rho^2}{4}\right) z^3 + \frac{\rho^2}{4} z^4 \ , \quad \Phi(z) = \rho(1-z)$$

whose Hawking temperature is

$$T = \frac{1}{4\pi}\left(3 - \frac{\rho^2}{4}\right)$$

$\boxed{m^2 = -2\;,\;\Delta = 2}$

To find the critical temperature T_c, we need to solve the field equation for Ψ.

At extremality,

$$\frac{\rho^2}{4} = 3 \;\;, \quad T = 0$$

The general *real* solution to the field equation is

$$\Psi = A\frac{z}{z-z_+}\left(\frac{z-z_-}{z-z_+}\right)^{\frac{2\sqrt{2}-i}{2\sqrt{3}}q}\left(\frac{1-z}{z-z_+}\right)^{\delta_+} F\left(-\delta_- + 2\sqrt{\frac{2}{3}}q, -\delta_- - \frac{iq}{\sqrt{3}}; -2\delta_-; 2z_+^2\frac{1-z}{z-z_+}\right) + \text{c.c.}$$

where $z_\pm$ are the roots of $f(z)$ that are distinct from the horizon ($z_\pm \neq 1$) and

$$\delta_\pm = -\frac{1}{2} \pm i\frac{\sqrt{1+4q^2}}{2\sqrt{3}}$$

are the dimensions of a scalar field in AdS$_2$ space of effective mass

$$m_{\text{eff}}^2 = m^2 + g^{tt}q^2\Phi^2 = -2 - 2q^2$$

The radius of the AdS$_2$ space is $R = 1/\sqrt{6}$.

Since both dimensions are complex, neither solution is regular at the horizon.

- in accord with the fact that we are below the BF bound in AdS$_2$ space.

Near extremality,

$$\frac{\rho^2}{4} = 3 - 4\pi T_c \ , \quad T_c \ll 1$$

Above $T = 0$ result still approximately valid away from the horizon.

Near the horizon, perform the coordinate transformation

$$z = 1 - \frac{2\pi T_c}{3}\zeta$$

The metric becomes

$$ds^2 = \frac{1}{6\zeta^2}\left[-(4\pi T_c)^2\zeta(1+\zeta)dt^2 + \frac{d\zeta^2}{\zeta(1+\zeta)}\right] + d\vec{x}^2 + \dots$$

where we omitted higher-order terms in T_c.

The electrostatic potential reads

$$\Phi = \frac{2\pi\rho T_c}{3}\zeta + \dots$$

and the field equation for the scalar field Ψ near the horizon becomes

$$\zeta(1+\zeta)\Psi'' + (2\zeta+1)\,\Psi' + \frac{1}{3}\left[1 + q^2\frac{\zeta}{1+\zeta}\right]\Psi = 0$$

where prime denotes differentiation with respect to ζ.

The acceptable solution is

$$\Psi(\zeta) = (1+\zeta)^{-iq/\sqrt{3}}\,F\left(-\delta_+ - \frac{iq}{\sqrt{3}}, -\delta_- - \frac{iq}{\sqrt{3}}; 1; -\zeta\right)$$

arbitrarily normalized so that at the horizon $\Psi(0) = 1$. The other solution is discarded because it has a logarithmic singularity at the horizon.

It is easily deduced from the identity

$$F(\alpha,\beta;\gamma;x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta;\gamma;x)$$

that Ψ is real.

As $\zeta \to \infty$, we obtain

$$\Psi \approx \frac{\Gamma(-1-2\delta_+)}{\Gamma(-\delta_+ + \frac{iq}{\sqrt{3}})\Gamma(-\delta_+ - \frac{iq}{\sqrt{3}})}\zeta^{\delta_-} + \text{c.c.}$$

to be compared with the near horizon limit of the other expression,

$$\Psi \sim A(1-z_+)^{\delta_-}\left(\frac{1-z_-}{1-z_+}\right)^{\frac{2\sqrt{2}-i}{2\sqrt{3}}q}(1-z)^{\delta_+}+\text{c.c.}$$

We obtain

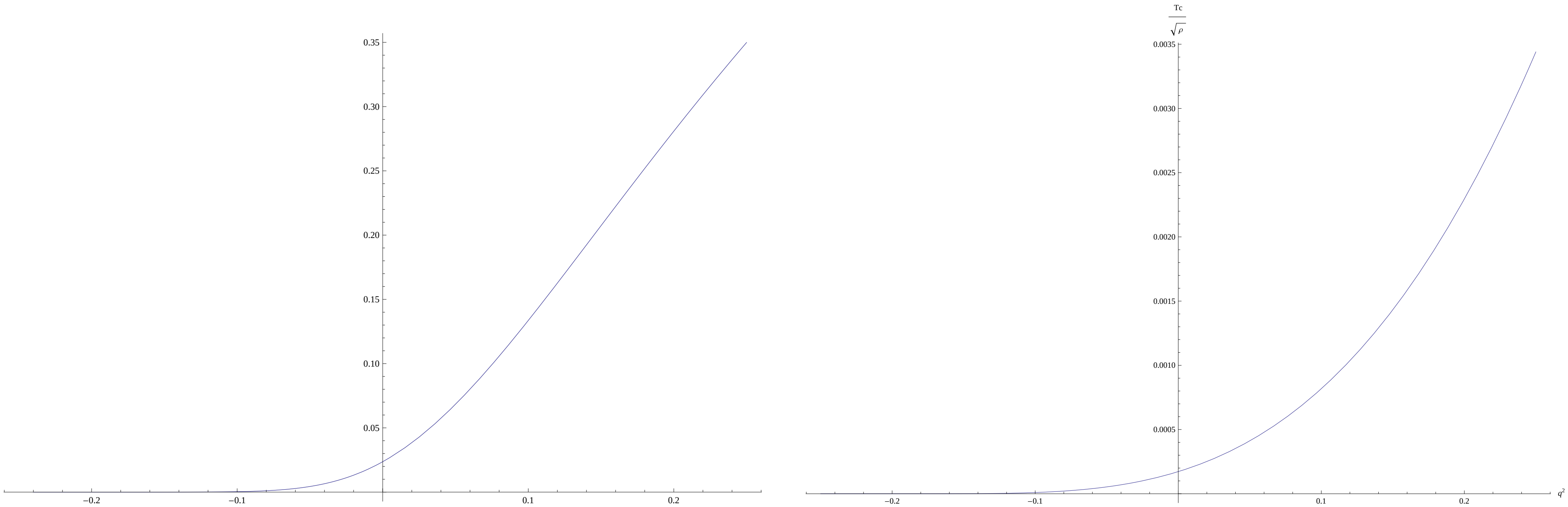
$$A=\frac{\Gamma(-1-2\delta_-)}{\Gamma(-\delta_-+\frac{iq}{\sqrt{3}})\Gamma(-\delta_--\frac{iq}{\sqrt{3}})}\left(\frac{2\pi T_c}{3}\right)^{-\delta_+}(1-z_+)^{-\delta_-}\left(\frac{1-z_+}{1-z_-}\right)^{\frac{2\sqrt{2}-i}{2\sqrt{3}}q}$$

At the boundary, we have $\Psi \sim z^2$, therefore

$$A(-z_+)^{\delta_-}\left(\frac{z_-}{z_+}\right)^{\frac{2\sqrt{2}-i}{2\sqrt{3}}q}F\left(-\delta_-+2\sqrt{\frac{2}{3}}q,-\delta_--\frac{iq}{\sqrt{3}};-2\delta_-;-2z_+\right)+\text{c.c.}=0$$

Solve this constraint $\Rightarrow \quad T_c$

$$T_c/\sqrt{\rho} \quad vs. \quad q^2$$



$\Delta = 1$ (left panel), $\Delta = 2$ (right panel).

- $T_c \longrightarrow 0$ as $q^2 \longrightarrow -\frac{1}{4}$.
- Meaning of negative q^2?

Write complex scalar in terms of real scalars as $\frac{1}{\sqrt{2}}\Psi e^{iq\theta}$. Then action

$$S_{scalar} = \frac{1}{2}\int d^4x\sqrt{-g}\left[\partial_\mu\Psi\partial^\mu\Psi + q^2\Psi^2(\partial_\mu\theta - A_\mu)^2 - m^2\Psi^2\right]$$

θ is Stückelberg field giving mass to A_μ when Ψ condenses.

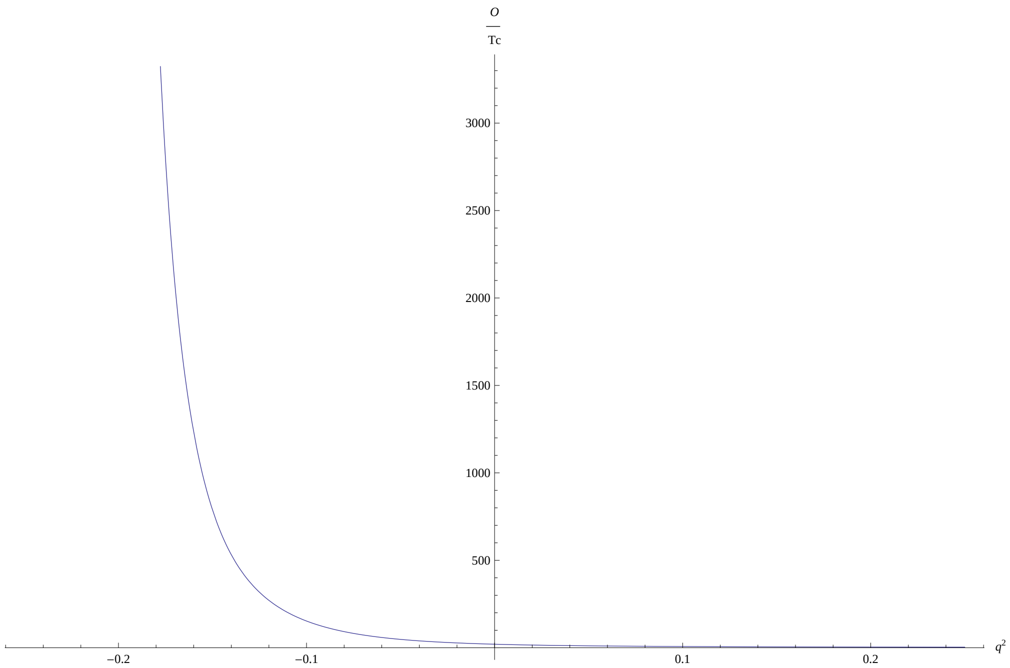
- ◇ fix gauge by setting $\theta = 0$
- ◇ $q^2 < 0 \ \Rightarrow$ negative mass - no instability in AdS above BF bound.
- ◇ $q = 0$: θ decouples. Physical consequences?

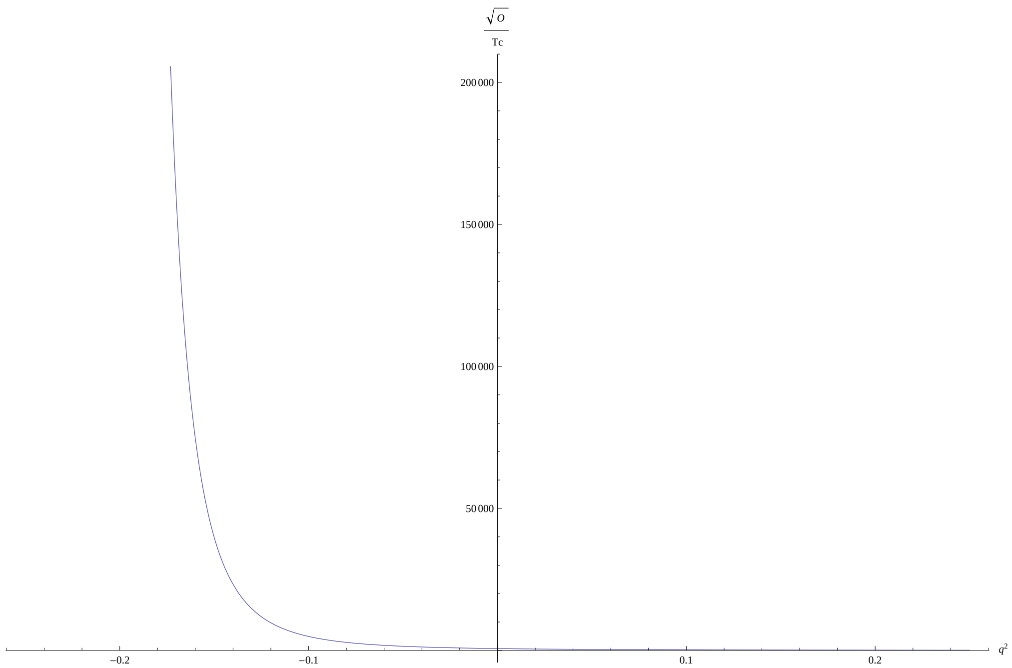
For general m,

$$q^2 \geq \frac{3 + 2m^2}{4}$$

Below T_c use iteration to solve field equations.

$$\langle\mathcal{O}\rangle^{1/\Delta}/T_c \quad vs. \quad q^2$$





$\Delta = 1$ (left panel), $\Delta = 2$ (right panel).

- Energy gap in units of T_c diverges as $T_c \to 0$.
- *cf.* with 3.4 (BCS) and ~ 8 as $q^2 \to \infty$.

Conclusions

- Gravity provides a *dual* description of superconductors at strong coupling.
- used analytic techniques to probe low temperature physics where numerical instabilities arise, in order to understand the *ground state* of these systems
- Found that, in the probe limit, the condensate diverges at the BF bound (albeit very mildly) and conductivity has an infinite number of poles whose positions we determined analytically.
- Found analytic relation between energy gap E_g and frequency gap in the conductivity ω_g which differs from BCS $E_g = \omega_g/2$.
- Developed analytic tools to probe possible quantum critical points (as one varies the charge q or the mass m of the scalar field). Found diverging E_g/T_c and $T_c \to 0$ as $q^2 \to \frac{3+2m^2}{4}$ (negative q^2!).
- A lot of work remains to be done toward a more realistic description, e.g., by incorporating spatial dependence to account for the lattice structure, doping, etc.