Quantum Foundations and Neutrinos

Based on arXiv:2310.07457 and planned work with **D. Minic** and **T. Takeuchi**

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The study of quantum foundations

- Early debates/studies on the foundations of QM mostly concerned its interpretations.
- Difficult to test with actual experiments.
- John Bell's work (1964) codified our intuition about the classical world in the form of inequalities which allowed us to look for their violations experimentally.
- QM, and any theory that isn't classical, violates Bell inequalities.
- This paved a path to study quantum foundations in a rigorous and experimentally testable way.

Where does our work fit in?

One way to study the foundations of QM is by generalizing its mathematical framework, which is the focus of this talk.

- Better understanding: Relaxing the mathematical structure and generalizing QM can give insights into the aspects that were generalized.
- New phenomenology: It could describe physical phenomena not present in canonical QM.
- More parameters $\Longrightarrow$ Wider testing: It could allow for a wider testing of certain aspects of QM.

Generalizations of quantum mechanics

- Canonical QM can be generalized in several distinct directions: Non-linear Schrödinger equation, replace $\mathbb{C}$ with $\mathbb{H}$, etc.
- QM has a rigid structure $\implies$ Changes in dynamics can have unphysical consequences.
- For example, Weinberg's non-linear QM allows for FTL communication!
- In general, new parameters that quantify the deviation from QM should be strongly constrained.
- Our work generalizes QM through its geometric formulation.[1]

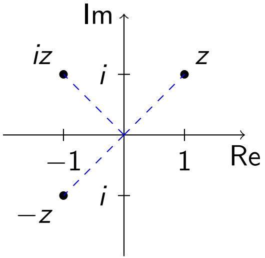
[1] This formulation was developed in Kibble ('79), Heslot ('85), Ashtekar ('97), Brody ('99) and more.

Geometric quantum mechanics: Structure I

- Vector space $\mathbb{C}^N$ = Vector space $\mathbb{R}^{2N}$ + Additional structure.
- To see that, write $\psi \in \mathbb{C}$ as a vector in $\mathbb{R}^2$:

$$\psi = \psi_\alpha + i\psi_\beta \to \vec{\psi} = \begin{bmatrix} \psi_\alpha \\ \psi_\beta \end{bmatrix}.$$

- Then $i \to J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $J^2 = -I$. $J\vec{\psi} = \begin{bmatrix} -\psi_\beta \\ \psi_\alpha \end{bmatrix}$.



Geometric quantum mechanics: Structure II

- The complex inner product between $\psi = \psi_\alpha + i\psi_\beta$ and $\phi = \phi_\alpha + i\phi_\beta$ is given by

$$\langle\psi|\phi\rangle = \psi^*\phi = \underbrace{(\psi_\alpha\phi_\alpha + \psi_\beta\phi_\beta)}_{\vec{\psi}\cdot\vec{\phi}} + i\underbrace{(\psi_\alpha\phi_\beta - \psi_\beta\phi_\alpha)}_{\vec{\psi}\times\vec{\phi}}.$$

- So the probability amplitude is given by

$$|\langle\psi|\phi\rangle|^2 = |\vec{\psi}\cdot\vec{\phi}|^2 + |\vec{\psi}\times\vec{\phi}|^2.$$

Geometric quantum mechanics: Dynamics

- Expand a state in terms of its energy eigenstates as $|\psi\rangle = \sum_n \psi_n |n\rangle$.

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \implies \psi_n = N_n \, e^{-i\omega_n t},$$

where $H |n\rangle = \hbar\omega_n |n\rangle$.

- Write $\psi_n = q_n + ip_n$. Then $\vec{\psi}(t) = N_n \begin{bmatrix} \cos(\omega_n t) \\ -\sin(\omega_n t) \end{bmatrix}$, and

$$\frac{dq_n}{dt} = \omega_n p_n, \quad \frac{dp_n}{dt} = -\omega_n q_n.$$

- These are the classical Hamilton equations for coupled harmonic oscillators!

$$H = \sum_n \frac{1}{2}\omega_n (q_n^2 + p_n^2).$$

Generalization of geometric quantum mechanics

A generalization suggests itself: Replace the dynamics of the harmonic oscillator with a more complicated Hamiltonian. But not every arbitrary extension will be consistent and physically sensible!

- We extend this dynamics to that of an asymmetric top, with two conserved quantities

$$E = \frac{q_1^2}{2I_1} + \frac{q_2^2}{2I_2} + \frac{q_3^2}{2I_3}, \text{ and } L^2 = q_1^2 + q_2^2 + q_3^2.$$

- The equations of motion are given by

$$\frac{dq_i}{dt} = \epsilon_{ijk} \left(\frac{1}{I_j} - \frac{1}{I_k} \right) q_j q_k.$$

Jacobi elliptic functions

- The solution of the above equations of motion is given in terms of Jacobi elliptic functions.

$$
\begin{aligned}
q_1(t) &= \quad N_1 \,\mathrm{cn}(\Omega t, k), \\
q_2(t) &= -N_2 \,\mathrm{sn}(\Omega t, k), \\
q_3(t) &= -N_3 \,\mathrm{dn}(\Omega t, k).
\end{aligned}
$$

- These functions appear when parametrizing the arc-length of an ellipse of eccentricity k.

$$
\begin{aligned}
\mathrm{sn}(u, k) &= \left(1 + \frac{k^2}{16} + \frac{7k^4}{256}\right) \sin v + \left(\frac{k^2}{16} + \frac{k^4}{32}\right) \sin(3v) + \cdots \\
\mathrm{cn}(u, k) &= \left(1 - \frac{k^2}{16} - \frac{9k^4}{256}\right) \cos v + \left(\frac{k^2}{16} + \frac{k^4}{32}\right) \cos(3v) + \cdots \\
\mathrm{dn}(u, k) &= \left(1 - \frac{k^2}{4} - \frac{5k^4}{64}\right) + \left(\frac{k^2}{4} + \frac{k^4}{16}\right) \cos(2v) + \cdots
\end{aligned}
$$

Consequences of the generalized dynamics

- The wavefunction is replaced by

$$\underbrace{\vec{\psi}_n \;=\; N_n \begin{bmatrix} \cos(\omega_n t) \\ -\sin(\omega_n t) \end{bmatrix}}_{|\vec{\psi}|^2 = N_n^2} \quad\rightarrow\quad \underbrace{\vec{\Psi}_n \;=\; A_n \begin{bmatrix} c_\xi\,\mathrm{cn}(\Omega_n t, k) \\ -\kappa_\xi\,\mathrm{sn}(\Omega_n t, k) \\ -s_\xi\,\mathrm{dn}(\Omega_n t, k) \end{bmatrix}}_{|\vec{\Psi}|^2 = A_n^2},$$

  where $c_\xi = \cos\xi,\ s_\xi = \sin\xi,\ \kappa_\xi = \sqrt{c_\xi^2 + k^2 s_\xi^2}$, and $0 \le k < 1$ and $-\frac{\pi}{2} \le \xi \le \frac{\pi}{2}$ are the deformation parameters.

- When $k = \xi = 0$, $\vec{\Psi}_n \rightarrow \vec{\psi}_n$ and the canonical QM limit is recovered.

- The inner product is generalized to

$$|\langle\Psi|\Phi\rangle|^2 \;=\; (\vec{\Psi}_n \cdot \vec{\Phi}_n)^2 + (\vec{\Psi}_n \times \vec{\Phi}_n)\cdot(\vec{\Psi}_n \times \vec{\Phi}_n).$$

- Generalized probability amplitude $\Longrightarrow$ Observable consequences!

Phase space of Nambu[2] quantum mechanics

The parameters $k \sim$ eccentricity and $\xi \sim$ size.

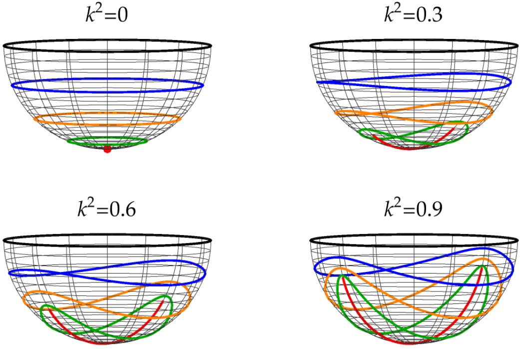


Figure: The colored lines in each figure indicate $\xi = 0$ (Black), $\xi = \pi/8$ (Blue), $\xi = \pi/4$ (Orange), $\xi = 3\pi/8$ (Green), and $\xi = \pi/2$ (Red). When $\xi = 0$, the trajectory always follows the equator regardless of the value of k.

[2] The reason for this name is explained in Minic & Tze, Phys.Lett.B 536 (2002) 305–314.

Neutrino oscillation probability

- Flavor eigenstates of neutrinos, $|\alpha\rangle$ and $|\beta\rangle$, are superpositions of their mass eigenstates, $|1\rangle$ and $|2\rangle$.

$$\begin{aligned} |\alpha\rangle &= \quad \cos\theta\,|1\rangle + \sin\theta\,|2\rangle \\ |\beta\rangle &= -\sin\theta\,|1\rangle + \cos\theta\,|2\rangle \end{aligned}$$

- This causes the phenomena of interference and oscillation.
- Neutrino oscillation is a function of the ratio $(\Delta t/E)$. For canonical QM,

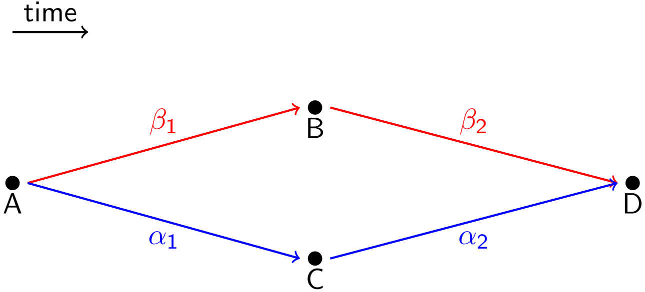
$$P(\alpha \to \beta) = \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 L}{4E}\right),\; L \approx c\Delta t.$$

For Nambu QM upto $\mathcal{O}(k^2)$,

$$P(\alpha \to \beta) = \left(c_\xi^2 + \frac{k^2}{2} s_\xi^2\right) \sin^2 2\theta\, \sin^2\left(\frac{\Delta m^2 L}{4E}\right).$$

Atmospheric neutrino data can be used to constrain these parameters. See arXiv:2310.07457.

Neutrino oscillation = Double slit experiment



$$P_{(\alpha,\alpha)}(A \to D) = |\alpha_1\alpha_2 + \beta_1\beta_2|^2$$

$$= \underbrace{|\alpha_1|^2 |\alpha_2|^2}_{P_{ACD}} + \underbrace{|\beta_1|^2 |\beta_2|^2}_{P_{ABD}} + \underbrace{2\,\mathrm{Re}(\alpha_1^*\alpha_2^*\,\beta_1\beta_2)}_{I_2(\alpha,\beta)}.$$

Legget-Garg inequality and neutrino oscillation

- Consider the two-flavor case again with flavors α and β. Let $Q = +1$ if a neutrino is found in flavor α and $Q = -1$ if in β.

$$\langle Q(t_i)Q(t_j)\rangle = \sum_{i,j} P_{ij} Q(t_i)Q(t_j).$$

- For classical theories

$$K_3 := \langle Q(t_0)Q(t_1)\rangle + \langle Q(t_1)Q(t_2)\rangle - \langle Q(t_0)Q(t_2)\rangle \leq 1.$$

- Quantum mechanics violates this Legget-Garg[3] inequality ($K_3 > 1$). Has been confirmed by tests on various atomic systems.
- Due to $(\Delta t/E)$ dependence, we can simply measure neutrinos at the same time but with different energies![4]

[3] A. J. Leggett and A. Garg PRL. 54, 857

[4] Formaggio et. al. Phys. Rev. Lett. 117, 050402

Testing LG with neutrino oscillations

- For two flavors,

$$\langle Q_i Q_j \rangle = \sum_{Q_i, Q_j = \pm 1} Q_i \, Q_j \, P_{ij}(t_i, t_j) = 2P_{\alpha\alpha}(t_i, t_j) - 1$$

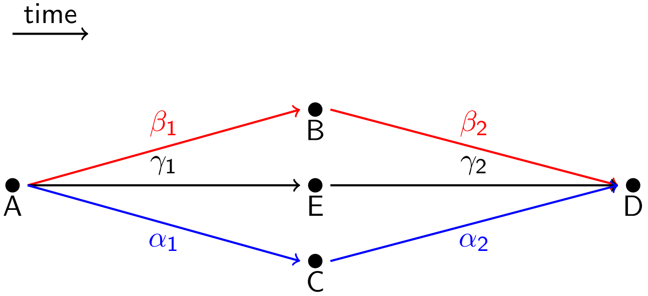
- Therefore,

$$K_3 = 1 - 4\,|\beta_1|^2 |\beta_2|^2 - \underbrace{4\,\mathrm{Re}(\alpha_1^* \alpha_2^* \beta_1 \beta_2)}_{2\,I(\alpha,\beta)}.$$ [5]

- Note that the term $2\,I(\alpha, \beta)$ makes $K_3 > 1$ possible.
- LG tests interference. That means it can be used to test or constrain a theory that predicts a different interference pattern than canonical QM.

[5] See D. S. Chattopadhyay and A Dighe (arXiv:2304.02475), where a different parameter is also proposed as a measure of "quantumness".

Triple path interference



- In a triple slit experiment:

$$P_{(\alpha,\alpha)}(A \to D) = |\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2|^2 = P_{ABD} + P_{AED} + P_{ACD} + I_2(\alpha,\beta) + I_2(\alpha,\gamma) + I_2(\beta,\gamma)$$

- The quantity $I_3(\alpha,\beta,\gamma)$ is non-zero only when there is triple-path interference.[6]

[6] Sorkin ('94) has studied this hierarchy in detail. Also see PRD 105, 115013 by PH, HM, DM, RP, TT for relation to neutrinos.

Phenomenology beyond quantum mechanics

- $\vec{\psi}_m \times \vec{\phi}_n$ calculates the area of the parallelogram spanned by the vectors $\vec{\psi}_m$ and $\vec{\phi}_n$. It gives rise to non-zero interference in QM.

- It might be possible to include a triple-interference term in Nambu QM if we can incorporate a volume element

$$\vec{\psi}_m \cdot (\vec{\phi}_n \times \vec{\chi}_k)$$

  into a unitary formulation.

- This in turn would modify LG and violate it more strongly than canonical QM.

Summary

- The foundations of quantum mechanics can be confronted with experiments.
- Generalizations of QM can help test old assumptions and provide new phenomenology.
- Nambu QM extension can potentially provide a model for two beyond QM phenomena: the triple path interference and super-quantum correlations.
- Neutrinos might turn out to be ideal systems for probing these issues experimentally.